

Solutions of massive gravity theories in constant scalar invariant geometries

K. Siampos^a, Ph. Spindel^b

Mécanique et Gravitation, Université de Mons, 7000 Mons, Belgique.

Synopsis

We solve massive gravity field equations for all locally homogenous and vanishing scalar invariant (VSI) Lorentzian spacetimes, which in three dimensions up to fibering and warping are the building blocks of constant scalar invariant (CSI) spacetimes. At first, we provide an exhaustive list of all Lorentzian 3D homogeneous spaces and then we determine the Petrov type of the relevant curvature tensors. Among these geometries we determine for which values of their structure constants they are solutions of the field equations of massive gravity theories with cosmological constant. The homogeneous solutions founded are of all various Petrov types : $I_{\mathbb{C}}$, $I_{\mathbb{R}}$, II , III , D_t , D_s , N , O ; the VSI geometries which we found are of Petrov type III .

^akonstantinos.siampos@umons.ac.be, ^bphilippe.spindel@umons.ac.be

Contents

| | |
|---|-----------|
| Contents | 1 |
| 1 Introduction | 1 |
| 2 Massive Gravity Theories | 3 |
| 2.1 Topologically massive gravity | 3 |
| 2.2 New massive gravity | 3 |
| 2.3 General Massive Gravity | 4 |
| 3 Homogeneous geometries | 4 |
| 3.1 Bianchi classification revisited | 5 |
| 3.2 Unimodular Lie algebra | 6 |
| 3.3 Non-unimodular Lie algebra | 8 |
| 4 Solutions of the field equations | 10 |
| 4.1 Simply transitive groups | 10 |
| 4.2 Coordinate representation of the metric | 22 |
| 4.3 Non simply transitive groups | 24 |
| 5 Vanishing scalar invariant geometries | 25 |
| 6 Conclusion | 29 |
| 7 Acknowledgments | 30 |
| A Petrov classification of homogeneous spaces | 30 |
| References | 38 |

1 Introduction

It is known that, in three dimensions, Einstein gravity theory does not possess any local physical degrees of freedom. However Deser, Jackiw and Templeton proposed a modification of the theory [1], consisting to add to the usual Einstein–Hilbert Lagrangian a Chern–Simons term [2, 3], defining so what is known as topological massive gravity (TMG). So they provided a chiral three-dimensional gravity theory that acquires massive spin-2 excitations and mediates finite-range interactions. This modification also proved to be an ideal example for a consistent quantum theory of gravity: it is unitary [4, 5] and super-renormalisable [6].

On the other hand Bergshoeff, Hohm and Townsend [7, 8] have recently obtained another type of massive gravity theory (NMG): a parity preserving theory that describes (on a Minkowski background) the propagation of a massive positive energy spin-2 field, but now of both helicities ± 2 . This theory possesses all the virtues of the TMG, so it may be considered as another consistent theory of quantum gravity. They have also considered the “merging” of both theories, producing a general massive gravity theory (GMG) which involve two spin-2 helicity states with different masses (parity-violating) and, as the previous ones, can be also extended by adding a cosmological constant.

Our work is motivated by the paper of Chow et al. [9] which provided a review of a large set of solutions of topological massive gravity (with cosmological constant). In their paper these authors provided a 3D variant of Petrov classification and showed that all the solutions they founded in the literature at this time are of Petrov type D or N , corresponding to locally squashed AdS_3 or AdS pp-wave metrics. Moreover they also proved that all Petrov type D solution of TMG actually are biaxially squashed AdS_3 metrics. Moreover, in a companion paper [10] these authors obtained new solutions of the topologically massive gravity equations by considering Kundt metrics [11, 12] [see also Chapters 28 and 31 of [13]]. These metric are characterised by the existence of a null geodesic vector field (that in 3D is automatically shear-free and twist-free). The TMG solutions belonging to this class of metric are generically of Petrov type II , but also in some special cases of Petrov types D , III , N and O . The classification of the homogeneous solutions to the classical vacuum equations of TMG were found by Ortiz in [14] and was recently generalised for non-vanishing cosmological constant [15].

On the other hand, it was proved in [16] that a Lorentzian 3D spacetime which is locally a constant curvature space (CSI) can be constructed, by means of fibering and warping, from locally homogeneous spaces and a subclass of CSI which are the vanishing scalar invariant spaces (VSI). This motivated our restriction to the geometries we considered in this work.

The paper is organised as follows: In section 2 we summarise the various field equations of the massive gravity theories. We start section 3 by revisiting all the three dimensional homogeneous spaces with Lorentzian signature and determine the Petrov type of the relevant curvature tensors (listed in the appendix), which will prove to be useful for solving the equations and identifying the solutions. We then proceed in section 4 to solve the equations of massive gravity theories for homogeneous spaces with cosmological constant. In the last section 5 we consider solutions of massive gravity theories for VSI geometries.

2 Massive Gravity Theories

Let us first remind the Einstein–Hilbert action,

$$S_{EH} = \frac{1}{\kappa^2} \int \sqrt{|g|} (R - 2\Lambda) d^3x \quad , \quad (2.1)$$

where Λ is the cosmological constant and κ is the gravitational coupling with mass dimension $[\kappa] = -\frac{1}{2}$. In what follows we adopt the mostly plus expression of the metric, define the curvature tensors so that the curvature of the Euclidean round sphere equipped with its positive definite metric as positive curvature¹ and fix the spacetime orientation by choosing $\varepsilon_{012} = +1$.

2.1 Topologically massive gravity

Topological massive gravity is obtained by adding a gravitational Chern–Simons term to the Einstein–Hilbert action

$$S_{TMG} := S_{EH} + \frac{1}{\mu \kappa^2} S_{CS} \quad , \quad (2.2)$$

$$S_{CS} = \frac{1}{2} \int \sqrt{|g|} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}{}^\rho \left(\partial_\mu \Gamma_{\rho\nu}{}^\sigma + \frac{2}{3} \Gamma_{\mu\tau}{}^\sigma \Gamma_{\nu\rho}{}^\tau \right) d^3x \quad ,$$

which is expressed through Christoffel symbols of the spacetime metric g while μ is a new coupling constant with mass dimension one.

The classical equations of motion read as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \quad , \quad (2.3)$$

$$C_{\mu\nu} := \frac{\varepsilon_\mu{}^{\rho\sigma}}{\sqrt{|g|}} \nabla_\rho \left(R_{\nu\sigma} - \frac{1}{4} g_{\nu\sigma} R \right)$$

where $C_{\mu\nu}$ is the Cotton–York tensor : a symmetric, traceless and divergenceless tensor. An immediate consequence of the equations of motion is that the traceless part of the Ricci tensor and the Cotton–York are proportional

$$S_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \quad , \quad S_{\mu\nu} := R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R \quad , \quad (2.4)$$

and accordingly of the same Petrov type.

2.2 New massive gravity

This theory is defined by adding a quadratic curvature term [7] to the Einstein–Hilbert action

$$S_{NMG} := S_{EH} - \frac{1}{\xi \kappa^2} S_{QC} \quad , \quad (2.5)$$

$$S_{QC} = \int \sqrt{|g|} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right) d^3x \quad ,$$

¹In others words according to the conventions : $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} + \dots$; $R_{\nu\sigma} = R^\rho{}_{\nu\rho\sigma}$.

where ξ is a coupling constant with mass dimension two. The corresponding equations of motion read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} - \frac{1}{2\xi}K_{\mu\nu} = 0 \quad , \quad (2.6)$$

$$K_{\mu\nu} = 2\nabla^2 R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R + \frac{9}{2}R R_{\mu\nu} - 8R_{\mu}{}^{\kappa}R_{\nu\kappa} + g_{\mu\nu}\left(3R_{\kappa\lambda}R^{\kappa\lambda} - \frac{1}{2}\nabla^2 R - \frac{13}{8}R^2\right) \quad ,$$

where $K_{\mu\nu}$ is a symmetric and divergenceless tensor whose trace coincides with Lagrangian density of S_{QC} (2.5), namely

$$K \equiv g^{\mu\nu}K_{\mu\nu} = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 \quad . \quad (2.7)$$

A consequence of the equations of motion is that traceless parts of the Ricci tensor and $K_{\mu\nu}$ tensor are proportional

$$S_{\mu\nu} = \frac{1}{2\xi}\hat{K}_{\mu\nu} \quad , \quad \hat{K}_{\mu\nu} := K_{\mu\nu} - \frac{1}{3}g_{\mu\nu}K \quad . \quad (2.8)$$

Thus, the Petrov type of the traceless part of the Ricci tensor and of $\hat{K}_{\mu\nu}$, have to coincide in the framework of the new massive gravity.

2.3 General Massive Gravity

This is defined by the merge of topological and new massive gravity theories

$$S_{GMG} := S_{EH} + \frac{1}{\mu\kappa^2}S_{CS} - \frac{1}{\xi\kappa^2}S_{QC} \quad , \quad (2.9)$$

and the corresponding field equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} - \frac{1}{2\xi}K_{\mu\nu} = 0 \quad . \quad (2.10)$$

To understand the field content of the latter equations, we have to linearise them around a fixed background. This was performed around Minkowsky and AdS_3 backgrounds in [7] and [17] respectively. In particular, it was shown in the latter that there are two massive spin-2 modes, which are stable if

$$\xi(\xi + 4\mu^2) \geq 2\Lambda\mu^2 \quad . \quad (2.11)$$

Thus, for pure NMG, when μ goes to infinity, AdS_3 is stable if $\xi \geq \frac{\Lambda}{2}$. In what follows we shall not restrict the sign of the coupling constants, but just discuss them according to the parameters appearing in the expressions of the metrics we obtain.

3 Homogeneous geometries

The strategy we adopt to obtain all homogeneous spaces, solutions of the massive gravity field equations was the one advocated a long time ago by Ozsváth [18]. We take as metric components the canonical

form

$$(\eta_{\alpha\beta}) = \text{diag.}(-1, 1, 1) \quad (3.1)$$

but start from arbitrary structure constants of the Lie algebra and fix them using only $Iso(1, 2)$ transformations that preserve the expression (3.1) of the metric. Our strategy, is to first review the classification so to use it in the field equations and then solve the resulting algebraic equations.

3.1 Bianchi classification revisited

As it is well known (see for example [13]) the Lie algebra of the (right) invariant fields

$$[\xi_\alpha, \xi_\beta] = -C_{\alpha\beta}^\gamma \xi_\gamma \quad , \quad (3.2)$$

whose structure constants satisfy the Jacobi identity $C_{\kappa\alpha}^\delta C_{\beta\gamma}^\kappa \varepsilon^{\alpha\beta\gamma} = 0$, allocates in two classes. The unimodular Lie algebras, such that $C_{\alpha\kappa}^\kappa = 0$ and the non-unimodular ones such that $\frac{1}{2}C_{\alpha\kappa}^\kappa = k_\alpha \neq 0$. For the three dimensional real algebras (first classified by Bianchi), their structure constants can be parametrised in terms of the vector components k_a and a symmetric tensor density² $n^{\alpha\beta}$ (see [13] and refs therein).

$$C_{\beta\gamma}^\alpha = \varepsilon_{\beta\gamma\zeta} n^{\zeta\alpha} + k_\beta \delta_\gamma^\alpha - k_\gamma \delta_\beta^\alpha \quad (3.3)$$

while the Jacobi identity reduces to the condition $k_\alpha n^{\alpha\beta} = 0$. Thus, the $Iso(1, 2)$ classification of the structure constants reduces to the classification of symmetric tensor densities, a problem solved in [19] (see also refs [20, 21]), annihilating a vector.

Chow et al. [9] suggested to classify solutions of TMG according to the Segre classification of the traceless Ricci tensor $S_{\alpha\beta}$, which is equivalent to the Petrov classification of the Cotton–York tensor but not necessarily in the frame work of NMG. Here after we present the various canonical forms of the Lie algebra obtained using $Iso(1, 2)$ transformations and the Segre–Petrov type of their traceless Ricci and Cotton–York tensors. On a practical level, to make this classification it is not necessarily to know exactly the eigenvalues of the tensor we want to classify when they are all different. So we have just compute the discriminant of the characteristic equation. If it is positive, we know that two eigenvalues are complex conjugate and one is real and so corresponds to the case $I_{\mathbb{C}}$ (see ref.[9] for the notations); if it is negative the three eigenvalues are real and distinct, corresponding to the case $I_{\mathbb{R}}$. It is only when it is zero, in which case we know that at least one eigenvalue is double, that further analysis is required to determine the Jordan form of the tensor. But the degeneracy of the eigenvalues greatly facilitates the analysis. In this case if the characteristic polynomial reduces to its cubic term $P(\lambda) = \lambda^3$, the tensor is of Petrov type O , N , or III ; otherwise when the roots are equal, but non vanishing, the tensor is of Petrov type D_s , D_t or II .

²In what follows we shall call them respectively *structure vector* and *structure tensor density*.

3.2 Unimodular Lie algebra

Four types of normal forms are possible :

- Type *I*

$$(n_I^{\alpha\beta}) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{pmatrix} , \quad (3.4)$$

this form is equivalent to $\text{diag.}(a, b+c, b-c)$, but the latter appears to be more easy to handle for solving the field equations. By changing the orientation (in which case μ also changes its sign), we always may assume that $a > 0$ if it is non zero, otherwise that $b > 0$ if non zero, etc.

Obviously this structure tensor density may correspond to any of the unimodular Bianchi types (*I*, *II*, *VI*₀, *VII*₀, *VIII*, *IX*) of real Lie algebras (see for instance Ortiz [14]).

The matrix component of Ricci tensor in the invariant frame, of the metric so defined reads

$$(R_{\alpha\beta}) = \frac{1}{2} \begin{pmatrix} a^2 - 4c^2 & 0 & 0 \\ 0 & a(a+2b) & -2(a+2b)c \\ 0 & -2(a+2b)c & a(a+2b) \end{pmatrix} , \quad (3.5)$$

and its trace is equal to $R = \frac{1}{2} (a^2 + 4ab + 4c^2)$. Accordingly the traceless part of this tensor is

$$(S_{\alpha\beta}) = \begin{pmatrix} \frac{2}{3} (a^2 + ab - 2c^2) & 0 & 0 \\ 0 & \frac{1}{3} (a^2 + ab - 2c^2) & -(a+2b)c \\ 0 & -(a+2b)c & \frac{1}{3} (a^2 + ab - 2c^2) \end{pmatrix} . \quad (3.6)$$

To determine the Petrov type of this tensor, we have to obtain the Jordan form of the matrix of components S_{β}^{α} . Fortunately, we have not to handle explicit solutions of the third degree characteristic polynomial

$$P(\lambda) := \det[\lambda \delta_{\beta}^{\alpha} - S_{\beta}^{\alpha}] \quad (3.7)$$

in the general case. The traceless condition implies that this cubic polynomial, always will be of the form : $P(\lambda) = \lambda^3 + p\lambda + q$. Its discriminant, which is defined as $\Delta := \frac{q^2}{4} + \frac{p^3}{27}$, partially fixes the number and the nature of the different roots of $P(\lambda) = 0$. If $\Delta > 0$ one root is real and the two others complex conjugate; if $\Delta < 0$ the three roots are real and distinct; if $\Delta = 0$, at least two roots are equal. Accordingly, when $\Delta > 0$ the traceless tensor will be of Petrov type *I*_C and when $\Delta < 0$ of Petrov type *I*_R. When $\Delta = 0$, the tensor is of special Petrov type. It is of type *II* or *D* when $p \neq 0$, and of type *III*, *N* or *O* when $P(\lambda) = \lambda^3$. Its precise determination will need more investigation, but things are greatly facilitated as at least one of the eigenvalue of the tensor is degenerate. For example, in case of the structure tensor density (3.6), we obtain

$$\Delta_S = -\frac{1}{27} (a+2b)^2 c^2 ((a+b)^2 - c^2)^2 (a^2 - 4c^2)^2 , \quad (3.8)$$

which generically is negative and thus define a tensor of type $I_{\mathbb{R}}$; exceptions occur when it vanishes.

The same discussion can be performed on the Cotton–York tensor

$$(C_{\beta}^{\alpha}) = \begin{pmatrix} -a^2(a+b) - 4bc^2 & 0 & 0 \\ 0 & \frac{1}{2}(a^3 + ba^2 + 4bc^2) & c(2c^2 - \frac{1}{2}a^2 + 4b^2 + 2ab) \\ 0 & c(2c^2 - \frac{1}{2}a^2 + 4b^2 + 2ab) & \frac{1}{2}(a^3 + ba^2 + 4bc^2) \end{pmatrix}. \quad (3.9)$$

The results of this analysis, both for the $S_{\alpha\beta}$ with $C_{\alpha\beta}$ and $\widehat{K}_{\alpha\beta}$ tensors, are summarised in tables 1-6. Thus, solutions of TMG or NMG can be easily found by comparing the Petrov classifications of $S_{\alpha\beta}$ with $C_{\alpha\beta}$ or $\widehat{K}_{\alpha\beta}$ respectively. However, this is not *a priori* the case for GMG.

- Type *II*

$$(n_{II}^{\alpha\beta}) = \begin{pmatrix} \nu + a & \nu & 0 \\ \nu & \nu - a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \nu = \pm 1 \quad (3.10)$$

which does not admit timelike eigenvector but a double null vector.

Diagonalising n_{II} with a $GL(3, \mathbb{R})$ transformation we see that if $a = b = 0$, this corresponds to a Bianchi *II* type of Lie algebra, if $b = 0$ but $a \neq 0$ or $a = 0$ and $\nu b < 0$ to Bianchi VI_0 , if $a = 0$ and $\nu b > 0$ to Bianchi VII_0 , and otherwise to Bianchi *VIII*.

The Ricci and Cotton–York tensors read

$$(R_{\alpha\beta}) = \frac{1}{2} \begin{pmatrix} -(2a+b)(b-2\nu) & -2(2a+b)\nu & 0 \\ -2(2a+b)\nu & (2a+b)(b+2\nu) & 0 \\ 0 & 0 & -b^2 \end{pmatrix}, \quad (3.11)$$

$$(C_{\beta}^{\alpha}) = \frac{1}{2} \begin{pmatrix} -((8a^2 + 4ba - b^2)\nu + b^2(a+b)) & (8a^2 + 4ba - b^2)\nu & 0 \\ -(8a^2 + 4ba - b^2)\nu & (8\nu a^2 - b(b-4\nu)a - b^2(b+\nu)) & 0 \\ 0 & 0 & 2b^2(a+b) \end{pmatrix}. \quad (3.12)$$

- Type *III*

$$(n_{III}^{\alpha\beta}) = \begin{pmatrix} a & 1 & 0 \\ 1 & -a & 1 \\ 0 & 1 & -a \end{pmatrix}, \quad (3.13)$$

which admits a triple null vector.

This structure constant density may only corresponds to Lie algebra of Bianchi types VI_0 if $a = 0$ and for $a \neq 0$ it is *VIII*.

The Ricci and Cotton–York tensors are

$$(R_{\alpha\beta}) = \begin{pmatrix} \frac{1}{2}a^2 - 2 & -a & 2 \\ -a & -\frac{1}{2}a^2 & a \\ 2 & a & -\frac{1}{2}a^2 - 2 \end{pmatrix}, \quad (3.14)$$

$$(C_{\beta}^{\alpha}) = \begin{pmatrix} 6a & \frac{3}{2}a^2 & -6a \\ -\frac{3}{2}a^2 & 0 & \frac{3}{2}a^2 \\ 6a & \frac{3}{2}a^2 & -6a \end{pmatrix}. \quad (3.15)$$

- Type *IV*

$$(n_{IV}^{\alpha\beta}) = \begin{pmatrix} 0 & \nu & 0 \\ \nu & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad (3.16)$$

where $a^2 < 4\nu^2$ and which only has one simple spacelike eigenvector, but no timelike or null eigenvector.

Here again, only the structure constants of Lie algebras of Bianchi types VI_0 if $b = 0$ and for $b \neq 0$ it is $VIII$. The corresponding Ricci and Cotton–York tensors read

$$(R_{\alpha\beta}) = \frac{1}{2} \begin{pmatrix} -(a-b)^2 & 2(a-b)\nu & 0 \\ 2(a-b)\nu & b^2 - a^2 & 0 \\ 0 & 0 & a^2 - b^2 - 4\nu^2 \end{pmatrix}, \quad (3.17)$$

$$(C_{\beta}^{\alpha}) = \begin{pmatrix} -\frac{1}{2}(a-b)(a^2 - b^2) & \nu(2(a^2 - \nu^2) - \frac{1}{2}(a+b)^2) & 0 \\ -\nu(2(a^2 - \nu^2) - \frac{1}{2}(a+b)^2) & a(a^2 - 2\nu^2) - \frac{1}{2}b(a^2 + b^2) & 0 \\ 0 & 0 & a(2\nu^2 - \frac{1}{2}a^2) + b^2(b - \frac{1}{2}a) \end{pmatrix}. \quad (3.18)$$

3.3 Non-unimodular Lie algebra

Here we have to consider three possibilities: the vector of components k_a is timelike, spacelike or null; four type of normal forms will occur.

- Timelike k_{α} : We choose the frame such that $k_{\alpha} = (k, 0, 0)$. The Jacobi identity implies that $n^{\alpha\beta}$ is a spacelike symmetric tensor that can be diagonalised by rotation in the $[1, 2]$ plane. But this normal form is not the most suitable for the resolution of the field equations and we prefer to use the following form

$$(n_T^{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}. \quad (3.19)$$

Obviously, all non-unimodular Lie algebra may lead to this form of the structure tensor density. More precisely we have a Lie algebra of Bianchi type V if $a = b = 0$, of type IV if $a = \pm b \neq 0$, and otherwise of types III , VI_h for $|b| > |a|$ or VII_h for $|a| > |b|$ with $h = k^2/(a^2 - b^2)$.

The corresponding Ricci and Cotton–York tensors read

$$(R_{\alpha\beta}) = 2 \begin{pmatrix} -(b^2 + k^2) & 0 & 0 \\ 0 & k(k+b) & -ab \\ 0 & -ab & k(k-b) \end{pmatrix}, \quad (3.20)$$

$$(C_{\beta}^{\alpha}) = 2 \begin{pmatrix} -2ab^2 & 0 & 0 \\ 0 & ab(b-3k) & b(2a^2 + b^2 - k^2) \\ 0 & b(2a^2 + b^2 - k^2) & ab(b+3k) \end{pmatrix}. \quad (3.21)$$

- Spacelike k_α : We choose the frame such that $k_\alpha = (0, 0, k)$. Then the structure tensor density may take three different canonical forms according to that it admits a timelike (and thus a spacelike) eigenvector

$$(n_{SI}^{\alpha\beta}) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad (3.22)$$

corresponding to any type B Bianchi space, namely: Bianchi type V if $a = b = 0$, type IV if a or b non-zero, and otherwise type III , VI_h for $ab < 0$ or VII_h for $ab > 0$ with $h = k^2/(ab)$. If it has a double null eigenvector :

$$(n_{SII}^{\alpha\beta}) = \begin{pmatrix} \nu + a & \nu & 0 \\ \nu & \nu - a & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \nu = \pm 1 \quad ; \quad (3.23)$$

corresponding to Lie algebra of Bianchi type IV if $a = 0$, otherwise to Bianchi type III and VI_h with $h = -k^2/a^2$. If the structure tensor density did not have any other eigenvector, it can be put in the form :

$$(n_{SIII}^{\alpha\beta}) = \begin{pmatrix} 0 & \nu & 0 \\ \nu & a & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad a^2 < 4\nu^2 . \quad (3.24)$$

and corresponds to Bianchi types III and VI_h with $h = -k^2/\nu^2$. In SI case, the Ricci and Cotton–York tensors read

$$(R_{\alpha\beta}) = \frac{1}{2} \begin{pmatrix} a^2 - b^2 + 4k^2 & -2k(a+b) & 0 \\ -2k(a+b) & a^2 - b^2 - 4k^2 & 0 \\ 0 & 0 & (a+b)^2 - 4k^2 \end{pmatrix} , \quad (3.25)$$

$$(C_{\beta}^{\alpha}) = \frac{1}{2} \begin{pmatrix} -(a+b)(2a^2 - ab + b^2 + 2k^2) & 3k(a^2 - b^2) & 0 \\ -3k(a^2 - b^2) & (a+b)(a^2 - ab + 2(b^2 + k^2)) & 0 \\ 0 & 0 & (a+b)(a^2 - b^2) \end{pmatrix} . \quad (3.26)$$

in case SII

$$(R_{\alpha\beta}) = 2 \begin{pmatrix} k^2 + (k+a)\nu & -\nu(a+k) & 0 \\ -\nu(a+k) & \nu(a+k) - k^2 & 0 \\ 0 & 0 & -k^2 \end{pmatrix} , \quad (3.27)$$

$$(C_{\beta}^{\alpha}) = -2\nu(a+k)(2a+k) \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.28)$$

and in case $SIII$

$$(R_{\alpha\beta}) = \begin{pmatrix} 2k(k+\nu) - \frac{1}{2}a^2 & a(\nu-k) & 0 \\ a(\nu-k) & -\frac{1}{2}a^2 - 2k(k-\nu) & 0 \\ 0 & 0 & \frac{1}{2}a^2 - 2(k^2 + \nu^2) \end{pmatrix} , \quad (3.29)$$

$$(C_{\beta}^{\alpha}) = \begin{pmatrix} -a(\frac{1}{2}a^2 + k(k-3\nu)) & \frac{1}{2}(k-\nu)(4\nu(k+\nu) - 3a^2) & 0 \\ -\frac{1}{2}(k-\nu)(4\nu(k+\nu) - 3a^2) & a(a^2 + k^2 - 3k\nu - 2\nu^2) & 0 \\ 0 & 0 & a(2\nu^2 - \frac{1}{2}a^2) \end{pmatrix}. \quad (3.30)$$

- Lightlike k_{α} : Here, without lost of generality, we may assume $k_{\alpha} = (1, 1, 0)$. Using the Jacobi identity we obtain the expression

$$(n_L^{\alpha\beta}) = \begin{pmatrix} a & -a & b \\ -a & a & -b \\ b & -b & c \end{pmatrix} \quad (3.31)$$

corresponding to Bianchi type *V* if $a = b = c = 0$, type *IV* if $b^2 = ac$ and Bianchi type *III*, *VI_h* for $b^2 > ac$ or *VII_h* for $b^2 < ac$ with $h = 1/(ac - b^2)$ in the other cases. Without lost of generality, it can still be simplified, by performing an appropriate null rotation around k_a , that leads to

$$(n_L^{\alpha\beta}) = \begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & c \end{pmatrix} \quad \text{or} \quad (n_{L'}^{\alpha\beta}) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & -b \\ b & -b & 0 \end{pmatrix}. \quad (3.32)$$

The Ricci and Cotton–York tensors are

$$(R_{\alpha\beta}) = \begin{pmatrix} c(a - \frac{1}{2}c) - 2b(1+b) & ac - 2b(1+b) & (1+b)c \\ ac - 2b(1+b) & c(a + \frac{1}{2}c) - 2b(1+b) & (1+b)c \\ (1+b)c & (1+b)c & -\frac{1}{2}c^2 \end{pmatrix}, \quad (3.33)$$

$$(C_{\beta}^{\alpha}) = \begin{pmatrix} -c(2b^2 + 3b + 1 + \frac{1}{2}c(c-a)) & -c(2b^2 + 3b + 1 - \frac{1}{2}ac) & \frac{3}{2}(1+b)c^2 \\ c(2b^2 + 3b + 1 - \frac{1}{2}ac) & c(2b^2 + 3b + 1 + \frac{1}{2}c(c+a)) & -\frac{3}{2}(1+b)c^2 \\ -\frac{3}{2}(1+b)c^2 & -\frac{3}{2}(1+b)c^2 & c^3 \end{pmatrix}. \quad (3.34)$$

4 Solutions of the field equations

4.1 Simply transitive groups

In the framework of the homogeneous spaces, whose group of isometries is strictly transitive (Bianchi spaces), the field equations reduce to algebraic equations. Our strategy in solving the equations, is to first obtain the link between the cosmological and structure constants Λ and μ (*resp.* ξ) and the structure constant parameters and then inserting them into the field equations and discuss the remaining constraints that have to be satisfied.

We shall provide some details in the first case, whereas in the other cases we shall just present the solutions of the equations.

- Unimodular Lie algebra

Type I

★ TMG: Using the expression (3.4) of the tensor density defining the structure constants, the TMG field equations reduces to three independent equations

$$2a^2(a+b) + 8bc^2 + (a^2 - 4c^2 + 4\Lambda)\mu = 0 \quad , \quad (4.1)$$

$$\frac{a^2}{4} + ab + c^2 - 3\Lambda = 0 \quad , \quad (4.2)$$

$$c(-a^2 + 4ab + 8b^2 + 4c^2 - 2(a+2b)\mu) = 0 \quad , \quad (4.3)$$

which are easy to solve.

First let us assume $c = 0$, so that eq. (4.3) is trivially satisfied. We deduce from the other two that:

$$\Lambda = \frac{1}{12}a(a+4b) \quad , \quad (4.4)$$

$$\mu = -\frac{3}{2}a \quad , \quad (4.5)$$

from which we obtain the solution of Petrov type D_t :

$$a = -\frac{2}{3}\mu \quad , \quad b = \frac{\mu^2 - 27\Lambda}{6\mu} \quad , \quad c = 0 \quad . \quad (4.6)$$

To isolate the value (4.5) of μ in eq. (4.1) we have assumed that $a^2 + 4\Lambda \neq 0$. If $a^2 + 4\Lambda = 0$, the parameter μ remains undetermined and the field equations are satisfied if the cosmological constant is negative and

$$a = -b = 2\sqrt{-\Lambda} \quad , \quad (4.7)$$

or if $\Lambda = 0$, for $a = 0$ and b arbitrary. Of course these last two solutions are conformally flat (Petrov type O). More precisely if $a = 0$ the solution is flat; otherwise it is AdS_3 if $a = -b \neq 0$.

If we assume $c \neq 0$, the values of Λ and μ provided by the first two field equations (4.1),(4.2) become

$$\Lambda = \frac{1}{12}(a^2 + 4ab + 4c^2) \quad , \quad (4.8)$$

$$\mu = -\frac{3}{2} \left(\frac{a^2(a+b) + 4bc^2}{a(a+b) - 2c^2} \right) \quad . \quad (4.9)$$

Inserting these values into the third non trivial field equation (4.3) we obtain

$$((a+b)^2 - c^2)(a^2 + 4ab + 4c^2) = 0 \quad , \quad (4.10)$$

as we assume of course that $\mu \neq 0$. This leads to two solutions (changing the sign of c correspond to a reflexion of θ^1 or θ^2 and thus to change the sign of μ) of Petrov type D_s :

$$a = \frac{27\Lambda - \mu^2}{6\mu} \quad , \quad b = \frac{5\mu^2 - 27\Lambda}{12\mu} \quad , \quad c = \pm(a+b) = \pm \frac{\mu^2 + 9\Lambda}{4\mu} \quad (4.11)$$

and a third one, of Petrov type $I_{\mathbb{R}}$, non flat, but with vanishing cosmological constant and

$$b = \frac{1}{2}(\mu + a) \quad , \quad c^2 = -\frac{1}{4}a(3a + 2\mu) \quad , \quad (4.12)$$

which restrict the parameter a to belongs to the interval³: $a \in [0, -\frac{2}{3}\mu]$.

★ NMG: The strategy is the same, but the equations a little bit more cumbersome. The field equations lead to

$$-21a^4 - 20a^3b + 80abc^2 + 256b^2c^2 + 80c^4 + 8\xi(a^2 + 2ab - 4\Lambda) = 0 \quad , \quad (4.13)$$

$$-63a^4 - 80a^3b + 8a^2(-2b^2 + 3c^2 + 2\xi^2) + 16(16b^2c^2 + 5c^4 - 4c^2\xi + 4\Lambda\xi^2) = 0 \quad (4.14)$$

$$c(5a^3 - 2a^2b + 40ab^2 - 8\xi(a + 2b) + 20ac^2 + 64b^3 + 104bc^2) = 0 \quad . \quad (4.15)$$

Here also three different types of solutions occur.

If we assume $c = 0$ then we obtain

$$\Lambda = \frac{a(21a^2 + 72ab + 16b^2)}{8(21a + 4b)} \quad , \quad (4.16)$$

$$\xi = \frac{a}{8}(21a + 4b) \quad , \quad (4.17)$$

which leads to a solution of Petrov type D_t :

$$a^2 = \frac{16}{21} \left(3\xi \pm \sqrt{3}\sqrt{\xi(7\Lambda + 5\xi)} \right) \quad , \quad (4.18)$$

$$b^2 = \frac{42\Lambda\xi \pm \sqrt{3}(21\Lambda - 17\xi)\sqrt{\xi(7\Lambda + 5\xi)} - 66\xi^2}{4(\Lambda - \xi)} \quad . \quad (4.19)$$

We find more convenient to discuss this solution by introducing the parameter $x = b/a$ in terms of which we may rewrite eqs (4.16), (4.17) as

$$\Lambda = \frac{a^2}{8} \frac{21 + 72x + 16x^2}{21 + 4x} \quad , \quad \xi = \frac{a^2}{8} (21 + 4x) \quad . \quad (4.20)$$

In order to have $\xi > 0$ we need, in addition to $a \neq 0$, that $x > -21/4$, which implies that we always have $\Lambda > 0$ if $-(2\sqrt{15} + 9)/4 > x > -21/4$ or $x > (2\sqrt{15} - 9)/4$.

If we did not assume $c = 0$, the first two equations (4.13), (4.14) lead to much more complicated expressions

$$\Lambda = \frac{2ac^2(15a^3 + 28a^2b - 128b^3) + a^3(a+b)(21a^2 + 72ab + 16b^2) - 16bc^4(15a + 32b) - 160c^6}{8(a^2(a+b)(21a + 4b) - 2c^2(3a^2 + 10ab + 64b^2) - 40c^4)} \quad (4.21)$$

$$\xi = \frac{a^2(a+b)(21a + 4b) - 2c^2(3a^2 + 10ab + 64b^2) - 40c^4}{16(a(a+b) - 2c^2)} \quad , \quad (4.22)$$

³When we write an interval as $[x_0, x_1]$ we did not assume $x_0 \leq x_1$, but consider the union of the sets $\{x|x_0 \leq x \leq x_1\} \cup \{x|x_1 \leq x \leq x_0\}$ even, unless $x_0 = x_1$, one of these two sets is always empty. This convention allows to avoid tedious (but elementary) discussion about signs.

inserting these in (4.15) field equation we find

$$a(a+b)^2(a^2+2ab-4b^2)-(a^3+10a^2b+12ab^2+8b^3)c^28bc^4=0 \quad (4.23)$$

whose solutions are $c^2=(a+b)^2=a^2(1+x)^2$ and $c^2=a(a^2+2ab-4b^2)/8b=a^2(1+2x-4x^2)/8x$.

From the first solution, which is Petrov type D_s , we obtain

$$\xi = \frac{a^2}{16}(1+2x)(25+42x) \quad , \quad \Lambda = \frac{a^2}{8} \frac{(1+2x)(84x^2+228x+109)}{(25+42x)} \quad , \quad (4.24)$$

which satisfy the requirement $\xi > 0$ if $x \notin [-25/42, -1/2]$. We can see immediately that this restriction is compatible with both signs of the cosmological constant : $\Lambda < 0$ if $x \in]-(8\sqrt{15}+57)/42, (8\sqrt{15}-57)/42[\cup]-25/42, -1/2[$, $\Lambda = 0$ at the boundary of these intervals, excepted at $x = -25/42$ where it diverges; otherwise $\Lambda > 0$ if $x \notin [-(8\sqrt{15}+57)/42, (8\sqrt{15}-57)/42] \cup]-25/42, -1/2]$. Thus here Λ is bounded from below (by approximatively $-0.3a^2$), but it can be chosen arbitrarily positive.

For the second solution $c^2=a^2(1+2x-4x^2)/8x$ things are a little bit more involved. We obtain

$$\xi = \frac{a^2}{32x}(64x^3-44x^2+36x+5) \quad , \quad \Lambda = \frac{5a^2}{16x} \frac{(1+2x)^4}{(64x^3-44x^2+36x+5)} \quad . \quad (4.25)$$

Thus ξ will be positive only if $x \notin [x_0, 0]$ where $x_0 \simeq -0.119$ is the single real root of the cubic polynomial $64x^3-44x^2+36x+5$. For $x \in [x_0, 0]$ we always have $\Lambda < 0$; for $x \notin [x_0, 0]$ we have $\Lambda > 0$, a decreasing function of x which varies from $+\infty$ to $5a^2/64$.

★ GMG: Combining the two theories introduce three constants that are related to the components of the structure tensor density (3.4) by

$$\Lambda = -\frac{5(a^2+4ab+4c^2)^3}{8(a^2(3a^2-40ab-112b^2)-8(9a^2+20ab-16b^2)c^2-80c^4)} \quad (4.26)$$

$$\mu = \frac{a^2(-3a^2+40ab+112b^2)+8(9a^2+20ab-16b^2)c^2+80c^4}{16(a^3+2a^2b-4ab^2-8bc^2)} \quad (4.27)$$

$$\xi = \frac{a^2(-3a^2+40ab+112b^2)+8(9a^2+20ab-16b^2)c^2+80c^4}{8(a^2+4ab+4c^2)} \quad (4.28)$$

Inverting this system as well as discussing in general the positivity of ξ is not very illuminating. For this last purpose we have plotted the region in the $(x = b/a, y = c/a)$ plane of the sign of ξ on fig. 1. Let us also remark that we always have $\Lambda \xi = (5/64)(a^2+4ab+4c^2)^2 \geq 0$.

However, using the special values of the structure tensor density we can easily see that there is no solution of type O which allows to have $\xi > 0$; while those of type D_t and D_s are both possible. As for a structure tensor density of type I , we have as many parameters in the solution as the physical constants of the problem. Thus we may expect that for some range of values a finite number of solutions are always defined. Of course, when the number of geometrical parameters will be less than three, the solutions, if any, will exist only for special values of the physical constants.

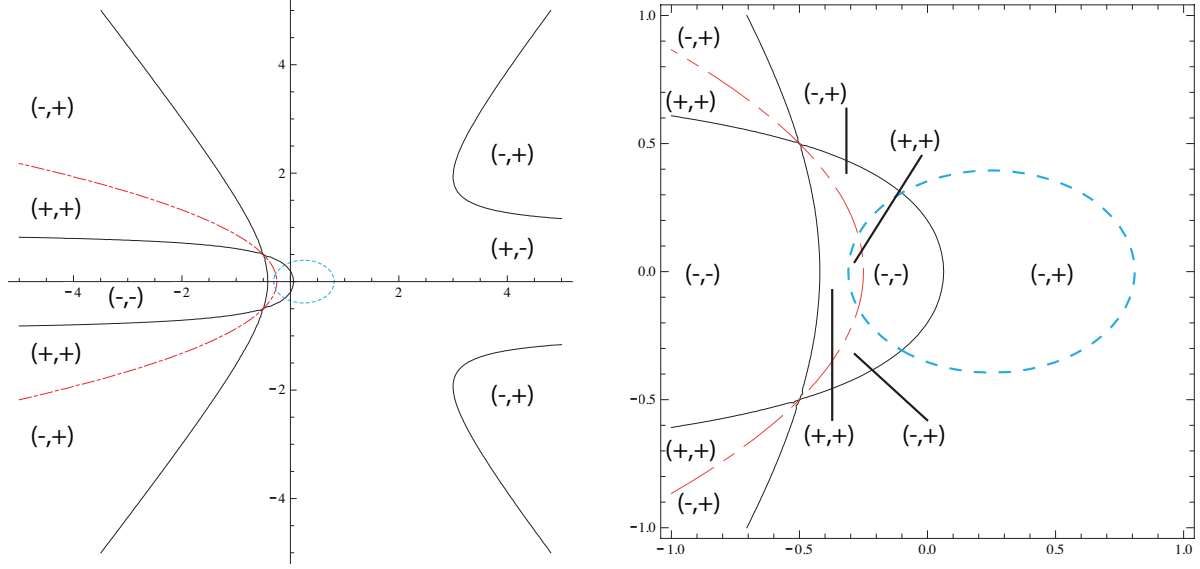


Figure 1: Graphical representation of the sign of the GMG coupling constant Λ and μ as function of the rescaled parameters $x = b/a$ and $y = c/a$ in the framework of unimodular group spaces of type *I*. We have $\Lambda = \infty$, $\xi = 0$, $\mu = 0$ on the solid curve; $\xi = \infty$, $\Lambda = 0$ on the dash-dotted curve; $\mu = \infty$ on the dotted curve. On each region of the x, y plane delimited by these curves we have indicated the sign of Λ (equal to the one of ξ) and the sign of μ . The left-hand side of the figure is a blow-up of the central part of the right-hand side plot.

Type II

★ TMG: Three different solutions occur. One, of Petrov type *N*, with negative cosmological constant

$$\Lambda = -\frac{\mu^2}{9} \quad , \quad a = -\frac{2\mu}{3} \quad , \quad b = \frac{2\mu}{3} \quad ; \quad (4.29)$$

and two, of Petrov type *II*, with vanishing cosmological constant

$$\Lambda = 0 \quad , \quad a = -\frac{\mu}{2} \quad , \quad b = 0 \quad ; \quad (4.30)$$

$$\Lambda = 0 \quad , \quad a = -\frac{\mu}{6} \quad , \quad b = \frac{2\mu}{3} \quad . \quad (4.31)$$

Let us mention that trivial flat solutions also occurs during the discussion of the field equations, for instance we meet the solution with $a = b = 0$, corresponding to the flat space⁴; we shall not insist anymore on such solutions.

★ NMG: We also obtain three types of solutions; two of Petrov type *N*:

$$b = 0 \quad , \quad a^2 = \frac{1}{4} \xi \quad , \quad \Lambda = 0 \quad (4.32)$$

$$b = -a \quad , \quad a^2 = \frac{8}{17} \xi \quad , \quad \Lambda = -\frac{35}{289} \xi \quad (4.33)$$

⁴This emphasises the obvious fact that a Bianchi *I* is a solution, always corresponds to a flat space, flat space may also appear as a Bianchi *III* model.

and a third one of Petrov type *II* :

$$b = -(1 \pm \sqrt{5})a \quad , \quad a^2 = \frac{(61 \mp 19\sqrt{5})}{479} \xi \quad , \quad \Lambda = \frac{5(139 \mp 1560\sqrt{5})}{229441} \xi \quad . \quad (4.34)$$

Let us noticed that the cosmological constant as well as the coupling constant are independent of ν , but not the curvature of the spaces.

★ GMG: We obtain :

$$\Lambda = -\frac{5b(4a+b)^3}{8(-112a^2 - 40ab + 3b^2)} \quad , \quad (4.35)$$

$$\mu = \frac{b(-112a^2 - 40ab + 3b^2)}{16(-4a^2 + 2ab + b^2)} \quad , \quad (4.36)$$

$$\xi = \frac{b(112a^2 + 40ab - 3b^2)}{8(4a+b)} \quad , \quad (4.37)$$

where $\Lambda\xi = (5/32)b^2(4a+b)^2 \geq 0$. To have $\xi > 0$, b must be in the intervals $]a(20 + 4\sqrt{46})/3, 0[\cup]a(20 - 4\sqrt{46})/3, -4a[$. These solutions always are of type *II*.

Solutions of type *N* also occur in the cases where $b = 0$

$$\Lambda = 0 \quad , \quad \xi = \frac{4a^2\mu}{2a+\mu} \quad , \quad (4.38)$$

or when $b = -a$

$$\Lambda = -a^2 \frac{70\mu + 3a}{272\mu} \quad , \quad \xi = \frac{17a^2\mu}{4(3a+2\mu)} \quad . \quad (4.39)$$

Type *III*

★ TMG: The condition $\mu \neq 0$ is incompatible with the field equations.

★ NMG: There is no solution with $\xi \neq 0$.

★ GMG: There is no solution with $\xi > 0$ but there is a special one of Petrov type *III* with $\xi \leq 0$

$$a = -\frac{80}{69}\mu \quad , \quad \Lambda = -\frac{4000}{12167}\mu^2 \leq 0 \quad , \quad \xi = -\frac{800}{207}\mu^2 \leq 0 \quad . \quad (4.40)$$

Type *IV*

★ TMG: Taking into account the condition $4\nu^2 > a^2$, we obtain as only (real) solution

$$a = \frac{\mu}{2} \mp \nu \quad , \quad b = \frac{\mu}{2} \pm \nu \quad , \quad \Lambda = 0 \quad (4.41)$$

of Petrov type I_C , and subject to the condition : $\nu \notin [\mp\mu/2, \pm\mu/6]$.

★ NMG: Once Λ and ξ are expressed in terms of a , b and ν , the remaining field equation are satisfied if

$$\nu^2 = b(a-b) \quad \text{or} \quad \nu^2 = \frac{(a^2 - b^2)(a-b)}{4a} \quad . \quad (4.42)$$

It is easy to verify that the first is disallowed by the condition $4\nu^2 > a^2$. Whereas for the second one, we may parametrise again the solutions as follows :

$$b = x a \quad , \quad (4.43)$$

$$\nu^2 = \frac{a^2}{4} (1 - x^2)(1 + x) \quad , \quad (4.44)$$

$$\Lambda = \frac{5 a^2}{8} \frac{(1 - x)^4 x^2}{(8 + 11 x + 18 x^2 - 5 x^3)} \quad , \quad (4.45)$$

$$\xi = \frac{a^2}{8} (8 + 11 x + 18 x^2 - 5 x^3) \quad . \quad (4.46)$$

The condition $4\nu^2 > a^2$ implies that $x > (1 + \sqrt{5})/2$ or $0 > x > (1 - \sqrt{5})/2$ while the positivity of ξ requires $x < x_0$ where x_0 is the single real root of the cubic polynomial $8 + 11x + 18x^2 - 5x^3$, *i.e.* $x_0 = \left(18 + \sqrt[3]{12987 - 60\sqrt{14370}} + \sqrt[3]{3(4329 + 20\sqrt{14370})}\right)/15$, $x_0 \approx 4.21$, and insures that $\Lambda > 0$. This solution always is of Petrov type I_C , both with respect to the Ricci and the Cotton–York tensors.

★ GMG: The generic solution is of type I_C with

$$\Lambda = -\frac{5 [(a - b)^2 - 4\nu^2]^3}{8 [(a - b)^2 (3a^2 + 26ab + 3b^2) + 8\nu^2(a - 9b)(a - b) - 80\nu^4]} \quad , \quad (4.47)$$

$$\mu = \frac{(a - b)^2 (3a^2 + 26ab + 3b^2) + 8\nu^2(a - 9b)(a - b) - 80\nu^4}{16 [(a - b)^2(a + b) - 4a\nu^2]} \quad , \quad (4.48)$$

$$\xi = -\frac{(a - b)^2 (3a^2 + 26ab + 3b^2) + 8\nu^2(a - 9b)(a - b) - 80\nu^4}{8 [(a - b)^2 - 4\nu^2]} \quad , \quad (4.49)$$

where $\Lambda \xi = (5/64)((a - b)^2 - 4\nu^2)^2 \geq 0$. We have plot on fig. 2 the zero and singular curves of Λ , μ and ξ on the $(x = a/\nu, y = b/\nu)$ plane. We also find a Petrov type D_s solution

$$a = b = -\frac{\xi}{2\mu} \quad , \quad \nu^2 = -\frac{2}{5}\xi > 0 \quad , \quad \text{with : } \Lambda = \frac{1}{5}\xi \quad . \quad (4.50)$$

• Non-unimodular Lie Algebra

Type T

★ TMG: We obtain two solutions, only defined for positive cosmological constant. The first one is of Petrov O and locally a dS_3 space,

$$b = 0 \quad , \quad k^2 = \Lambda \quad , \quad (4.51)$$

while the second one is of Petrov Type D_s

$$a = \frac{\mu}{3} \quad , \quad k^2 = \frac{3}{4}\Lambda - \frac{1}{36}\mu^2 \quad , \quad b^2 = \frac{3}{4}\Lambda + \frac{1}{12}\mu^2 \quad , \quad \Lambda > \mu^2/27 \quad . \quad (4.52)$$

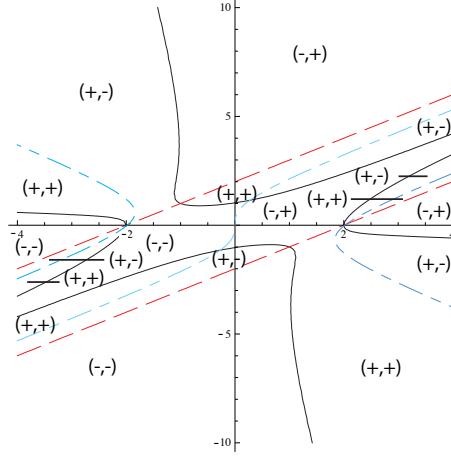


Figure 2: Graphical representation of the sign of the GMG coupling constant Λ and μ as function of the rescaled parameters $x = a/\nu$ and $y = b/\nu$ in the framework of unimodular group spaces of type IV. We have $\Lambda = \infty$, $\xi = 0$, $\mu = 0$ on the solid curve; $\xi = \infty$, $\Lambda = 0$ on the two dotted straight lines; $\mu = \infty$ on the dash-dotted curve. On each region of the x, y plane delimited by these curves we have indicated the sign of Λ (equal to the one of ξ) and the sign of μ .

★ NMG: Three different solutions are plausible. The first one is of Petrov type $I_{\mathbb{R}}$ (or Petrov type D_s when $\xi = 2\Lambda$)

$$a = 0 \quad , \quad b^2 = \frac{1}{8}(3\xi + \Lambda) \quad , \quad k^2 = \frac{1}{8}(5\Lambda - \xi) \quad (4.53)$$

which of course requires that $\Lambda > \xi/5$.

The second one is

$$b = 0 \quad , \quad k^2 = 2\left(\xi \pm \sqrt{\xi(\xi - \Lambda)}\right) \quad . \quad (4.54)$$

Note that for $\Lambda > 0$, k^2 has solution for the plus sign with $\xi \notin]0, \Lambda[$, whereas for $\Lambda < 0$ both signs in k^2 are accepted with $\xi > 0$.

The third one, of Petrov type D_s :

$$b^2 = a^2 + k^2 \quad , \quad a^2 = \frac{6\xi - \sqrt{\xi(15\xi + 42\Lambda)}}{42} \quad , \quad k^2 = \frac{\sqrt{\xi(15\xi + 42\Lambda)} - 4\xi}{8} \quad (4.55)$$

that are real only when $42\Lambda > \xi \geq 2\Lambda > 0$.

★ GMG: Generically we obtain solutions of Petrov type $I_{\mathbb{R}}$:

$$\Lambda = \frac{3k^4 - 14b^2k^2 + 16a^2k^2 - 5b^4}{2(k^2 + 8a^2 - 5b^2)} \quad , \quad (4.56)$$

$$\mu = \frac{k^2 + 8a^2 - 5b^2}{8a} \quad , \quad (4.57)$$

$$\xi = \frac{-k^2 - 8a^2 + 5b^2}{2} \quad . \quad (4.58)$$

The singular and zero curves of Λ , μ and ξ in the $(x = a/k, y = b/k)$ plane are depicted on fig. 3.

Let us mention that in the region where $\xi > 0$ (the interior of the hyperbola) $\Lambda > 0$.

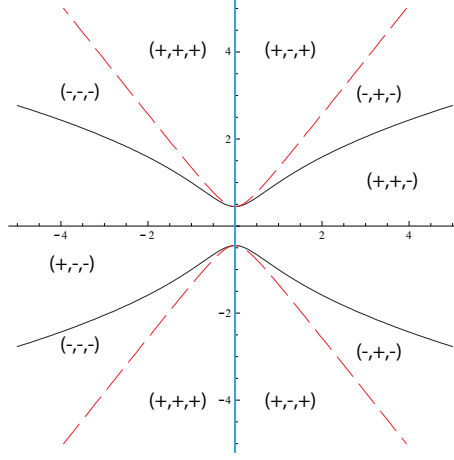


Figure 3: Graphical representation, in the framework of non-unimodular group spaces of type T , of the zero and singular curves of the cosmological constant Λ and of the coupling constants ξ and μ as function of the rescaled parameters $x = a/k$ and $y = b/k$ ($\Lambda = \infty$ but $\xi = 0$ and $\mu = 0$ on the (dashed) hyperbola, $\Lambda = 0$ on the black fourth order algebraic (solid) curve, $\mu = \infty$ on the y axis). For each region of the plane delimited by these curves, we indicate the sign of Λ , μ and ξ .

There also are solutions of Petrov type D_s , with $k^2 = b^2 - a^2$, μ arbitrary and

$$\Lambda = \frac{3a^5 - 35a^4\mu - 8a^3b^2 + 40a^2b^2\mu - 16ab^4 + 16b^4\mu}{2\mu(17a^2 + 4b^2)} \quad , \quad (4.59)$$

$$\xi = \mu \frac{17a^2 + 4b^2}{2(\mu - 3a)} \quad . \quad (4.60)$$

We plot on fig. 4 the zero curve of Λ and the singular straight line of ξ in the $(x = a/\mu, y = b/\mu)$ plane. Finally solutions of Petrov type O are obtained for arbitrary values of the coupling constant

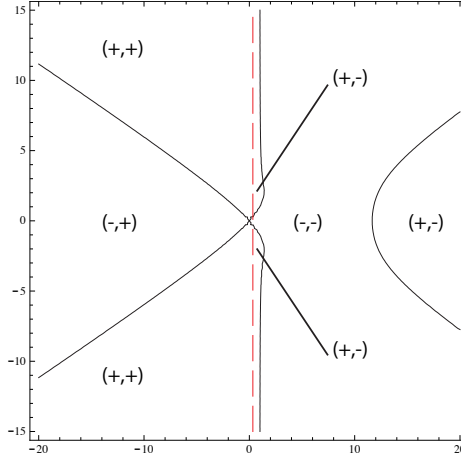


Figure 4: Graphical representation of the singular line ($x = 1/3$) of ξ and the zero curve of the cosmological constant Λ of the Petrov type D_s solution of GMG field equations, in the framework of non-unimodular group spaces of type T . The zero curve of Λ is a fifth order algebraic curve, admitting the vertical asymptote $x = 1$ and the two oblique asymptotes $y = \pm x/2$.

μ and ξ . They are dS_3 (or flat) spaces but with

$$\Lambda = \frac{4\xi - k^2}{4\xi} k^2 \quad . \quad (4.61)$$

Type SI

★ TMG: Two non trivial solutions occur. The first one, is Petrov O and locally an AdS_3 space with $\Lambda < 0$,

$$b = -a \quad , \quad k^2 = -\Lambda \quad . \quad (4.62)$$

The second one, with $\mu^2/27 > \Lambda \geq -\mu^2/9$, is of Petrov type D

$$a = \frac{-2\mu \pm \sqrt{27\Lambda + 3\mu^2}}{6} \quad , \quad b = \frac{2\mu \pm \sqrt{27\Lambda + 3\mu^2}}{6} \quad , \quad k^2 = -ab = \frac{\mu^2 - 27\Lambda}{36} \quad . \quad (4.63)$$

More precisely, with the upper sign (+), the solution is of Petrov type is D_s (*resp.* D_t) for $\mu > 0$ (*resp.* $\mu < 0$) ; with the lower sign (-), it is the converse : D_t (*resp.* D_s) for $\mu > 0$ (*resp.* $\mu < 0$).

★ NMG: Here we obtain $\xi = \frac{1}{8}[5(a+b)^2 + 16(a^2 + b^2) + 36k^2] > 0$ and the remaining field equations are satisfied in three cases.

If

$$a = -b \quad , \quad k^2 = 2 \left(-\xi \pm \sqrt{\xi(\xi - \Lambda)} \right) \quad , \quad (4.64)$$

which requires that $0 > \Lambda \geq \xi$ when k^2 is obtained with the minus sign in front of the square root or, if the plus sign is adopted : $\xi > 0$ and $\Lambda < 0$ or $\xi < 0$ and $\Lambda \geq \xi$. It is of Petrov type O , corresponding to an AdS_3 space for positive values of ξ .

We also obtain $k^2 = -ab$, in which case, parametrising the solution as before by $b = xa$, we have

$$\xi = \frac{a^2}{8}[8(x^2 + 1) + 13(x - 1)^2] > 0 \quad \text{and} \quad \Lambda = \frac{a^4}{64\xi}(21x^4 + 204x^3 - 194x^2 + 204x + 21) \quad . \quad (4.65)$$

The positivity of k^2 requires that $x < 0$. So the maximum of Λ/ξ is reached at $x = 0$ or $x = -\infty$ where the ratio tends to $1/21$. In the limit $x = 0$, we obtain $\Lambda = \xi/21 = a^2/8$. As the quartic polynomial $21x^4 + 204x^3 - 194x^2 + 204x + 21$ has only two (negative) real roots : $x_1 \approx -0.1$ and $x_2 \approx -10.7$, Λ is negative for the value of x between these two roots. It reaches its minimum at $x \approx -1$. The coupling constant ξ increases monotonically as x decreases; when x goes to $-\infty$, both Λ and ξ diverge. This solution is of Petrov type D_t if $x \in]-1, 0[$, Petrov type O when $x = -1$, and Petrov type D_s if $x \in]-\infty, -1[$.

There is also a third solution of Petrov type I_C (unless $a = b = 0$)

$$a = b = \pm \frac{\sqrt{\Lambda + 3\xi/2}}{2\sqrt{2}} \quad , \quad k^2 = \frac{\xi - 10\Lambda}{16} \quad , \quad (4.66)$$

according to sign of Λ we demand $\xi > 10\Lambda \geq 0$ or $\xi > -\frac{2}{3}\Lambda \geq 0$

★ GMG:

Generically we obtain a Petrov type I

$$\Lambda = \frac{-5(a+b)^4 - 8(a^2 - 30ab + b^2)k^2 + 48k^4}{8(3a^2 - 26ab + 3b^2 - 4k^2)} \quad , \quad (4.67)$$

$$\mu = \frac{-3a^2 + 26ab - 3b^2 + 4k^2}{16(a-b)} \quad , \quad (4.68)$$

$$\xi = \frac{1}{8}(-3a^2 + 26ab - 3b^2 + 4k^2) \quad . \quad (4.69)$$

We also recover a Petrov type O , namely an AdS_3 space when

$$b = -a \quad , \quad \Lambda = -k^2 \frac{k^2 + 4\xi}{4\xi} \quad . \quad (4.70)$$

We also obtain special solutions for $k^2 = -ab$, in which case we parametrise the solution as before by $b = xa$ with $x < 0$:

$$\Lambda = -a^2 \frac{(-21 - 55x + 14x^2 - 14x^3 + 55x^4 + 21x^5)\alpha - 2(21 + 204x - 194x^2 + 204x^3 + 21x^4)\mu}{16(21(1+x^2) - 26x)\mu} \quad ,$$

$$\xi = \frac{a^2\mu}{4} \frac{21(1+x^2) - 26x}{3(1-x)a + 2\mu} \quad . \quad (4.71)$$

This solution is of Petrov type D_t if $x \in]-1, 0[$, Petrov type O when $x = -1$, and Petrov type D_s if $x \in]-\infty, -1[$.

Type SII

★ TMG: We easily obtain a Petrov type N solution

$$k = \pm\sqrt{-\Lambda} \quad , \quad a = \frac{\mp\sqrt{-\Lambda} - \mu}{2} \quad ; \quad (4.72)$$

we also recover a Petrov type O , namely an AdS_3 space when

$$a = -k \quad , \quad \Lambda = -k^2 \quad . \quad (4.73)$$

★ NMG: Here, we obtain a Petrov type N . If

$$\Lambda = -\frac{k^2(4a+3k)(4a+k)}{2(8a^2+8ak+k^2)} \quad , \quad \text{and} \quad \xi = \frac{1}{2}(8a^2+8ak+k^2) \quad (4.74)$$

the field equations are satisfied. Accordingly, imposing $\xi > 0$, we obtain solutions for all values of a and k such that $k \notin [-4a - 2\sqrt{2}|a|, -4a + 2\sqrt{2}|a|]$. All these solution have negative cosmological constant. Let us notice that the value of the parameter ν did not play any rôle in the parametrisation of the solution, but in the curvature. For $a = -k$, we reobtain a Petrov type O , the AdS_3 geometry as a solution. For $a = -k/2$ we also have a conformally flat geometry, solving the field equations, but for negative value of $\xi = -2a^2$.

★ GMG:

Here, we obtain a Petrov type N

$$\Lambda = -\frac{16a^2\mu + k^2(k+3\mu) + 2ak(k+8\mu)}{2(8a^2 + 8ak + k^2)\mu}k^2, \quad \xi = \mu \frac{8a^2 + 8ak + k^2}{2(2a + k + \mu)}. \quad (4.75)$$

And a Petrov O type for $a = -k$, locally an AdS_3 space with $\Lambda = -\frac{k^2+4\xi}{4\xi}k^2$.

Type $S III$

★ TMG: We obtain a solution of Petrov type D :

$$a = \frac{2\mu}{3}, \quad k^2 = \nu^2 = \frac{\mu^2 - 27\Lambda}{36}, \quad \mu^2 < -9\Lambda, \quad (4.76)$$

which is Petrov type D_s (*resp.* D_t) for $k = \nu$ (*resp.* $k = -\nu$).

★ NMG: The following set of solutions occur:

Firstly a solution of Petrov type $I_{\mathbb{R}}$:

$$a = 0, \quad k^2 = \frac{\xi - 5\Lambda}{8}, \quad \nu^2 = -\frac{3\xi + \Lambda}{8}, \quad (4.77)$$

which imply that $\Lambda < -3\xi$.

Secondly, we also have solutions of Petrov type D :

$$k = \pm\nu, \quad a^2 = \frac{4}{21} \left(6\xi + \sqrt{3\xi(5\xi + 7\Lambda)} \right), \quad \nu^2 = \frac{1}{4} \left(4\xi + \sqrt{3\xi(5\xi + 7\Lambda)} \right) \quad (4.78)$$

which requires, taking into account the condition $4\nu^2 > a^2$ and assuming $\xi > 0$ that $\Lambda \geq -5\xi/7$, otherwise $2\Lambda \leq \xi < 0$.

Finally, we also obtain :

$$k = \pm\nu, \quad a^2 = \frac{4}{21} \left(6\xi - \sqrt{3\xi(5\xi + 7\Lambda)} \right), \quad \nu^2 = \frac{1}{4} \left(4\xi - \sqrt{3\xi(5\xi + 7\Lambda)} \right) \quad (4.79)$$

which requires that $0 > -35\xi/289 > \Lambda \geq -5\xi/7$. These solutions are of Petrov type D_s (*resp.* D_t) for $k = \nu$ (*resp.* $k = -\nu$).

★ GMG:

Here, we obtain solutions of Petrov type I

$$\Lambda = \frac{-5a^4 - 8a^2(5\nu^2 - k^2) + 16(3k^4 - 14k^2\nu^2 - 5\nu^4)}{8(3a^2 - 4k^2 + 20\nu^2)}, \quad (4.80)$$

$$\mu = \frac{3a^2 - 4k^2 + 20\nu^2}{16a}, \quad (4.81)$$

$$\xi = \frac{1}{8}(-3a^2 + 4(k^2 - 5\nu^2)). \quad (4.82)$$

We also find, solutions with $k = \pm\nu$ of Petrov type D

$$\Lambda = -\frac{21a^5 - 42a^4\mu - 160a^3\nu^2 + 576a^2\mu\nu^2 + 256a\nu^4 - 512\mu\nu^4}{336\mu a^2 - 256\mu\nu^2}, \quad (4.83)$$

$$\xi = \frac{-21a^2\mu + 16\mu\nu^2}{4(3a - 2\mu)}, \quad (4.84)$$

which are of Petrov type D_s (*resp.* D_t) for $k = \nu$ (*resp.* $k = -\nu$).

Type L The solutions obtained are of Petrov type *II*.

★ TMG: There is one solution, defined only for special values of μ and Λ (positive):

$$\mu = \pm 3\sqrt{3}\Lambda \quad , \quad a = -\frac{1}{c} \quad , \quad c = \pm 2\sqrt{3}\Lambda \quad . \quad (4.85)$$

★ NMG: Here also only one solution is obtained, for special values of ξ and Λ (positive):

$$\xi = 21\Lambda \quad , \quad a = -\frac{1}{c} \quad , \quad c = \pm 2\sqrt{2}\Lambda \quad . \quad (4.86)$$

★ GMG: There is a class of solutions for arbitrary a

$$\Lambda = -\frac{5}{24}c^2 \quad , \quad \mu = \frac{3}{16}c \quad , \quad \xi = -\frac{3}{8}c^2 \quad . \quad (4.87)$$

Type L' The solutions obtained are flat spaces.

★ TMG & NMG & GMG: The only solution is the flat space, obtained with

$$b = 0 \quad \text{or} \quad b = -1 \quad \text{and} \quad \Lambda = 0 \quad . \quad (4.88)$$

4.2 Coordinate representation of the metric

Having at our disposal all the homogeneous solutions of MG theories in a formal way, the purpose of this subsection is to perform an analytic example in which we express the invariant forms that define the coordinate system. The latter can be achieved algebraically by determining the $GL(3, \mathbb{R})$ matrix which transforms the canonical expression of the structure vector and tensor density into the forms we use and then apply the same transformation to the canonical expression of the invariant one-forms (for example see the table 8.2 of ref. [13]). Another possibility consists into a direct integration of the Cartan equations (3.2) defining the invariant vector fields from which we reduce the expression of their dual basis. For illustrative purpose we shall perform this calculation in the framework of the non-unimodular *SII* solution, which actually corresponds to the so-called pp-wave AdS [22, 15]] metric or null warped AdS metric.

The *SII* structure constants correspond to a group admitting an abelian two-dimensional subgroup. The corresponding Cartan equations are obtained from the structure tensor density (3.23) and the space like structure vector $k_\alpha = (0, 0, k)$. They read :

$$[\xi_1, \xi_2] = 0 \quad , \quad [\xi_1, \xi_3] = (k + \nu)\xi_1 + (\nu - a)\xi_2 \quad , \quad [\xi_2, \xi_3] = -(a + \nu)\xi_1 + (k - \nu)\xi_2 \quad . \quad (4.89)$$

Thanks to the presence of the abelian subgroup, the integration of these equations is immediate (see ref. [23] for a discussion of this problem in the framework of the Bianchi *III* group) and, after a specific

choice of the integration constants, leads to :

$$\xi_1 = \partial_x \quad , \quad \xi_2 = \partial_y \quad , \quad \xi_3 = \partial_z + r(x, y)\partial_x + s(x, y)\partial_y \quad . \quad (4.90)$$

where the function $r(x, y)$ and $s(x, y)$ are linear :

$$r(x, y) = (k + \nu)x - (a + \nu)y \quad \text{and} \quad s(x, y) = (\nu - a)x + (k - \nu)y \quad . \quad (4.91)$$

We immediately obtain the dual right invariant one-form that define the metric:

$$\theta^1 = dx - r(x, y) dz \quad , \quad \theta^2 = dy - s(x, y) dz \quad , \quad \theta^3 = dz \quad . \quad (4.92)$$

and the metric

$$ds^2 = -(dx - r(x, y) dz)^2 + (dy - s(x, y) dz)^2 + dz^2 \quad (4.93)$$

that solve the massive gravity equations when the algebraic conditions (4.72), (4.74), (4.75) are satisfied.

The metric (4.93) has an explicit the one-parameter (denoted λ_3) isometry subgroup :

$$z \mapsto z + \lambda_3 \quad (4.94)$$

but hides the two-parameter abelian isometry subgroup given by

$$x \mapsto x + p(z) \quad , \quad y \mapsto y + q(z) \quad , \quad z \mapsto z \quad (4.95)$$

where $p(z)$ and $q(z)$ are the solutions (depending on two arbitrary constants denoted hereafter λ_1 and λ_2 : the group parameters) of the differential system :

$$p'(z) = (k + \nu)p(z) - (a + \nu)q(z) \quad , \quad (4.96)$$

$$q'(z) = (\nu - a)p(z) + (k - \nu)q(z) \quad , \quad (4.97)$$

whose solution reads

$$p(z) = \lambda_1 e^{kz} (a \cosh(az) + \nu \sinh(az)) - \lambda_2 e^{kz} (\nu + a) \sinh(az) \quad (4.98)$$

$$:= \lambda_1 p_1(z) + \lambda_2 p_2(z) \quad , \quad (4.99)$$

$$q(z) = \lambda_1 e^{kz} (\nu - a) \sinh(az) + \lambda_2 e^{kz} (a \cosh(az) - \nu \sinh(az)) \quad (4.100)$$

$$:= \lambda_1 q_1(z) + \lambda_2 q_2(z) \quad . \quad (4.101)$$

Now it is immediate to write the most general left invariant vector, a Killing vector of the metric (4.93) depending on the three parameters λ_1, λ_2 and λ_3 introduced respectively in eqs (4.99) or (4.101) and (4.94) :

$$\tilde{\xi} = \lambda_3 \partial_z + p(z) \partial_x + q(z) \partial_y \quad . \quad (4.102)$$

To make explicit the action of the abelian two-dimensional subgroup we have to use the group parameters as coordinates. Assuming $a \neq 0$, we introduce new coordinates X and Y defined by

$$x = p_1(z) X + p_2(z) Y \quad , \quad (4.103)$$

$$y = q_1(z) X + q_2(z) Y \quad , \quad (4.104)$$

and we obtain

$$\begin{aligned} \theta^1 &= p_1(z) dX + p_2(z) dY \\ &= \frac{e^{kz}}{2} \{ e^{az} (\nu + a) (dX + dY) + e^{-az} ((a - \nu) dX + (a + \nu) dY) \} \quad , \end{aligned} \quad (4.105)$$

$$\begin{aligned} \theta^2 &= q_1(z) dX + q_2(z) dY \\ &= \frac{e^{kz}}{2} \{ e^{az} (\nu - a) (dX - dY) + e^{-az} ((a - \nu) dX + (a + \nu) dY) \} \quad . \end{aligned} \quad (4.106)$$

A last coordinate transformation :

$$X \mapsto \frac{\sigma \nu v + k^2 (\nu + a) u}{2 k^2 a \sqrt{|a|}} \quad , \quad Y \mapsto \frac{\sigma \nu v + k^2 (\nu - a) u}{2 k^2 a \sqrt{|a|}} \quad , \quad z \mapsto -\frac{1}{k} \ln(\zeta) \quad , \quad (4.107)$$

where $\sigma := -\text{sgn}(a) \nu$, provides the usual expression of the null warped AdS metric [15] :

$$ds^2 = \frac{1}{k^2} \frac{du dv + d\zeta^2}{\zeta^2} + \sigma \frac{du^2}{\zeta^{2(a/k+1)}} \quad . \quad (4.108)$$

Obviously the coordinate system (u, v, ζ) (or equivalently (x, y, z)) defines a local chart but does not cover the all group manifold.

4.3 Non simply transitive groups

The Kantowsky–Saks metric, who is the metric of an homogeneous space $\mathbb{R} \times S^2$, on which acts (multi-transitively) a 4-parameter isometry group that does not contain any 3-parameter subgroup is

$$ds^2 = -dt^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad . \quad (4.109)$$

This geometry is conformally flat, with a traceless Ricci tensor of Petrov type D_t , whereas the Cotton–York tensor is Petrov type O and the $\hat{K}_{\mu\nu}$ tensor is of Petrov type D_t . Thus, it can utmost satisfy the NMG (and so the GMG) field equations. It turns out that they are satisfied for $\Lambda = \frac{1}{2R^2}$, $\xi = \frac{1}{R^2}$. Let us emphasise that the hyperbolic or the flat version of this metric ($\sin \theta$ is replaced by $\sinh \theta$ or by θ respectively) admits a 3-parameter transitive isometry group. Thus, they have been already considered in the previous subsections, where indeed we find that the hyperbolic metric does not satisfy the TMG field equations but the equations of NMG (and GMG) are satisfied for $\Lambda = -\frac{1}{2R^2}$, $\xi = -\frac{1}{R^2}$.

5 Vanishing scalar invariant geometries

All the homogeneous geometries considered here above, share the common property, that all scalar geometrical objects which we can build are constants on the spacetime. Notably, in the frame of Lorentzian geometries there exist spaces (VSI spaces) which are not locally homogeneous but have vanishing scalar invariants built out of the curvature tensor. These geometries are a subclass of the Kundt geometries considered in [10] and explicitly known in three dimension [24]. If we exclude flat space, there are two possibilities such metrics⁵ (labelled **A1** and **B1** in ref. [24]) that all reads as follows

$$ds^2 = -2 du \left[dv + \frac{1}{2} F(u, v, x) du + W(u, v, x) dx \right] + dx^2 \quad (5.1)$$

with special expressions of the function $F(u, v, x)$ and $W(u, v, x)$. All these metric leads to a curvature (Ricci) tensor that can be expressed in a null frame $\{l, n, m\}$ such that $l^\alpha n_\alpha = -1 = -m^\alpha m_\alpha$ as

$$R_{\alpha\beta} = \phi l_\alpha l_\beta \quad \text{or} \quad R_{\alpha\beta} = \psi l_{(\alpha} m_{\beta)} \quad (5.2)$$

with $\phi \neq 0$ and $\psi \neq 0$. The first one is of Petrov type *N*, the second one of Petrov type *III*. Of course the functions appearing in the metric (5.1) can be modified by coordinate transformations which preserves its form:

$$u \mapsto \mathcal{U}[\tilde{u}] \quad , \quad v \mapsto \frac{\tilde{v}}{\dot{\mathcal{U}}[\tilde{u}]} + \mathcal{F}[\tilde{u}, \tilde{x}] \quad , \quad x \mapsto \tilde{x} + \mathcal{G}[\tilde{u}] \quad . \quad (5.3)$$

Using these transformations it is easy to check that the induced transformations on the functions (F, W) read

$$\tilde{F} = F \dot{\mathcal{U}}^2 + 2 \left(\dot{\mathcal{F}} \dot{\mathcal{U}} + W \dot{\mathcal{G}} \dot{\mathcal{U}} - \tilde{v} \frac{\ddot{\mathcal{U}}}{\dot{\mathcal{U}}} \right) - \dot{\mathcal{G}}^2 \quad , \quad \tilde{W} = W + \mathcal{F}' - \dot{\mathcal{G}} \quad , \quad (5.4)$$

where (F, W) are expressed in terms of the $(\tilde{u}, \tilde{v}, \tilde{x})$ coordinates via their (u, v, x) dependence given by eqs (5.3). Dot and prime denote partial derivatives with respect to the new coordinates \tilde{u} and \tilde{x} . In what follows we briefly describe the resolution of the massive gravity field equations in case of the VSI geometries. Let us notice that for consistency we have to assume the cosmological constant $\Lambda = 0$, as all scalar invariants of the metric considered vanish.

We have solved the equations of TMG, NMG and GMG for type **A1** and **B1**. It turns out that the only not trivial equations are those corresponding the $[u, x]$ and $[u, u]$ components. Our strategy is to first consider the simplest, i.e. $[u, x]$ equation and then plug its solution into the $[u, u]$ one and solve it.

- Type **A1** Here the metric components are a priori of the form

$$F(u, v, x) = v f_1(u, x) + f_0(u, x) \quad , \quad W(u, v, x) = w_0(u, x) \quad . \quad (5.5)$$

⁵In ref. [24] four non flat expressions of the metric are displayed, but the last two (denoted as D1&F1) are special cases of the first ones.

Before proceeding to the equations of motion we shall first discuss the gauge fixing which are allowed by the coordinate transformations (5.3). At first we note that we can always eliminate $w_0(u, x)$ with an appropriate choice of \mathcal{F}

$$\mathcal{F}(\tilde{u}, \tilde{x}) = - \int w_0(\tilde{u}, \tilde{x}) d\tilde{x} + \dot{\mathcal{G}}(\tilde{u})\tilde{x} + \mathcal{H}(\tilde{u}) \quad ; \quad (5.6)$$

thus we consistently assume that $w_0(u, x) = 0$. Using the latter, it easy to check that the residual coordinate transformations induce the transformations

$$\begin{aligned} \tilde{f}_1 &= f_1 \dot{\mathcal{U}} - 2 \frac{\ddot{\mathcal{U}}}{\dot{\mathcal{U}}} \quad , \\ \tilde{f}_0 &= f_0 \dot{\mathcal{U}}^2 + \left(\tilde{x} \dot{\mathcal{G}} + \mathcal{H} \right) f_1 \dot{\mathcal{U}}^2 + 2 \left(\tilde{x} \ddot{\mathcal{G}} + \dot{\mathcal{H}} \right) \dot{\mathcal{U}} - \dot{\mathcal{G}}^2 \quad . \end{aligned} \quad (5.7)$$

Here after we shall present the solution of the field equations in a coordinate system fixed by eliminating as many as possible gauge functions in the expression of the metric.

★ TMG The only non trivial field equations are $[u, x]$ and $[u, u]$. The first one gives

$$\partial_x^2 f_1(u, x) + \mu \partial_x f_1(u, x) = 0 \quad , \quad (5.8)$$

whose general solution reads

$$f_1(u, x) = c_0(u) + c_1(u) e^{-\mu x} \quad . \quad (5.9)$$

Using (5.7) we can eliminate $c_0(u)$ and set (locally) $c_1(u)$ to one by choosing $2\ddot{\mathcal{U}}/\dot{\mathcal{U}}^2 = c_0(u)$, $c_1(u) e^{-\mu \mathcal{G}} \dot{\mathcal{U}} = 1$, *i.e.* in Eq. (5.7) the $u = \mathcal{U}(\tilde{u})$ coordinate transformation as the inverse transformation of

$$\tilde{u} = \frac{1}{c_\star} \int e^{-\frac{1}{2} \int c_0(u) du} du \quad , \quad c_\star = \text{const.} \quad (5.10)$$

and with an appropriate sign for c_\star

$$\mathcal{G}(\tilde{u}) = \frac{1}{\mu} \left(\frac{1}{2} \int c_0(u) du + \ln[c_\star c_1(\mathcal{U}(\tilde{u}))] \right) \quad . \quad (5.11)$$

Let us notice that if $c_1(u) = 0$ the metric is flat. Using the latter solution, the $u u$ equation reduces to

$$\partial_x^3 f_0(u, x) + \mu \partial_x^2 f_0(u, x) = -\frac{\mu}{2} e^{-2\mu x} \quad . \quad (5.12)$$

whose general solution reads

$$f_0(u, x) = q_0(u) + q_1(u) x + q_2(u) e^{-\mu x} + \frac{e^{-2\mu x}}{8\mu^2} \quad , \quad (5.13)$$

Last but not least, we can also eliminate $q_2(u)$ with an appropriate choice of $\mathcal{H}(\tilde{u})$, *i.e.* $\mathcal{H}(\tilde{u}) = -q_2(\tilde{u})$.

★ NMG

Working similarly as in the TMG we obtain

$$f_1(u, x) = e^{-\sqrt{\xi}x} + c_2(u) e^{\sqrt{\xi}x} \quad (5.14)$$

and

$$\begin{aligned} f_0(u, x) = & q_0(u) + q_1(u)x + q_2(u) e^{-x\sqrt{\xi}} + \frac{2x\sqrt{\xi} - 5}{2\xi} c'_2(u) e^{x\sqrt{\xi}} \\ & + \frac{1}{6\xi} e^{-2x\sqrt{\xi}} + \frac{c_2^2(u)}{6\xi} e^{2x\sqrt{\xi}} \quad . \end{aligned} \quad (5.15)$$

★ GMG

Working similarly as in the TMG we obtain

$$f_1(u, x) = \begin{cases} e^{\lambda_- x} + c_2(u) e^{\lambda_+ x} , & \text{where } \lambda_{\pm} = \frac{\xi \pm \sqrt{\xi(\xi + 4\mu^2)}}{2\mu} , \\ c_2(u) x e^{-2\mu x} , & \text{if } \xi = -4\mu^2 \end{cases} \quad (5.16)$$

and

$$\begin{aligned} f_0(u, x) = & q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x))x + r_2(u, x)e^{\lambda_- x} + (q_3(u) + r_3(u, x))e^{\lambda_+ x} , \\ r'_0(u, x) = & \frac{1 + \mu x}{\xi \mu} j(u, x) , \quad r'_1(u, x) = -\frac{j(u, x)}{\xi} , \quad r'_{2|3}(u, x) = \mp \frac{e^{-x\lambda_{\mp}} j(u, x)}{\lambda_{\mp}^2 (\lambda_+ - \lambda_-)} , \end{aligned} \quad (5.17)$$

whereas for $\xi = -4\mu^2$

$$\begin{aligned} f_0(u, x) = & q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x))x + \{r_2(u, x) + (q_3(u) + r_3(u, x))x\} e^{-2\mu x} , \\ r'_0(u, x) = & -\frac{1 + \mu x}{4\mu^3} j(u, x) , \quad r'_1(u, x) = \frac{j(u, x)}{4\mu^2} , \quad r'_2(u, x) = \frac{(1 - \mu x) e^{2\mu x} j(u, x)}{4\mu^3} , \\ r'_3(u, x) = & \frac{e^{2\mu x} j(u, x)}{4\mu^2} , \end{aligned} \quad (5.18)$$

where

$$j(u, x) := \left((\partial_x f_1(u, x)) \partial_x + f_1(u, x) \left(\partial_x^2 - \frac{\xi}{2\mu} \partial_x \right) - \frac{\xi}{\mu} \partial_x \partial_u + 2\partial_u \partial_x^2 \right) f_1(u, x) \quad . \quad (5.19)$$

- Type **B1** The metric components are given by

$$F(u, v, x) = -\frac{v^2}{x^2} + v f_1(u, x) + f_0(u, x) \quad , \quad W(v, u, x) = -\frac{2v}{x} + w_0(u, x) \quad . \quad (5.20)$$

Applying the coordinate transformation (5.3) in our case, we find that the coordinate transformation which preserves the form of W is the v , for which we can always eliminate $w_0(u, x)$ as follows

$$\left(\partial_x - \frac{2}{x} \right) \mathcal{F}(u, x) = -w_0(u, x) \implies \mathcal{F}(u, x) = -x^2 \left(\int_x \frac{w_0(u, x')}{x'^2} dx' - \mathcal{H}(u) \right) \quad , \quad (5.21)$$

thus we can set consistently $w_0(u, x) = 0$. Using the latter, it easy to check that the residual coordinate transformations induce the transformations

$$\begin{aligned}\tilde{F} &= -\frac{\tilde{v}^2}{\tilde{x}^2} + \tilde{v} \tilde{f}_1 + \tilde{f}_0 \quad , \\ \tilde{f}_1 &= f_1 - 2\mathcal{H} \quad , \quad \tilde{f}_0 = f_0 + \tilde{x}^2 \mathcal{H}(f_1 - 1) + 2\tilde{x}^2 \dot{\mathcal{H}} \quad ,\end{aligned}\tag{5.22}$$

and as in Type **A1** we shall present the solutions of the field equations in a coordinate system fixed by eliminating as many as possible gauge functions in the expression of the metric.

★ TMG The only non trivial field equations are $[u, x]$ and $[u, u]$. The first one gives

$$\left(\partial_x^2 + \left(\mu + \frac{1}{x} \right) \partial_x \right) f_1(u, x) = 0 \quad ,\tag{5.23}$$

whose solution reads

$$f_1(u, x) = c_0(u) + c(u) Ei(-\mu x) \quad ,\tag{5.24}$$

where $Ei(z) = -\mathcal{PV} \int_{-z}^{+\infty} e^{-t}/t dt$, denotes the exponential integral function. Using (5.22) we can eliminate for example $c_0(u)$ by choosing $c_0(u) = 2\mathcal{H}(u)$. Using the latter equation, the $u u$ equation reads

$$\begin{aligned}\left(\partial_x^3 + \left(\mu - \frac{3}{x} \right) \left(\partial_x^2 - \frac{2}{x} \partial_x + \frac{2}{x^2} \right) \right) f_0(u, x) &= j(u, x) \quad , \\ j(u, x) &:= \frac{e^{-\mu x}}{2x} (c^2(u) Ei(-\mu x) + 2c'(u)) \quad ,\end{aligned}\tag{5.25}$$

whose solution can be found using the method of variation of parameters

$$\begin{aligned}f_0(u, x) &= x \left(q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x)) x + (q_2(u) + r_2(u, x)) e^{-\mu x} \right) \quad , \tag{5.26} \\ r'_0(u, x) &= -\frac{(1 + \mu x)j(u, x)}{\mu^2 x} \quad , \quad r'_1(u, x) = \frac{j(u, x)}{\mu x} \quad , \quad r'_2(u, x) = \frac{e^{\mu x} j(u, x)}{\mu^2 x} \quad .\end{aligned}$$

★ NMG

Working similarly as in the TMG we obtain

$$f_1(u, x) = c_1(u) Ei(-x\sqrt{\xi}) + c_2(u) Ei(x\sqrt{\xi})\tag{5.27}$$

and

$$\begin{aligned}f_0(u, x) &= x \left(q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x)) x + (q_2(u) + r_2(u, x)) e^{-x\sqrt{\xi}} + (q_3(u) + r_3(u, x)) e^{x\sqrt{\xi}} \right) \quad , \\ r'_0(u, x) &= \frac{j(u, x)}{x} \quad , \quad r'_1(u, x) = -\frac{j(u, x)}{x\xi} \quad , \quad r'_{2|3}(u, x) = \mp \frac{e^{\pm x\sqrt{\xi}} j(u, x)}{2x\xi^{3/2}} \quad .\end{aligned}\tag{5.28}$$

★ GMG

Working similarly as in the TMG we obtain

$$f_1(u, x) = \begin{cases} c_1(u) Ei(\lambda_- x) + c_2(u) Ei(\lambda_+ x) \quad , & \text{where } \lambda_{\pm} = \frac{\xi \pm \sqrt{\xi(\xi + 4\mu^2)}}{2\mu} \quad , \\ c_1(u) e^{-2\mu x} + c_2(u) Ei(-2\mu x) \quad , & \text{if } \xi = -4\mu^2 \end{cases}\tag{5.29}$$

and

$$f_0(u, x) = x (q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x))x + (q_2(u) + r_2(u, x))e^{\lambda_- x} + (q_3(u) + r_3(u, x))e^{\lambda_+ x}) ,$$

$$r'_0(u, x) = \frac{1 + \mu x}{\xi \mu x} j(u, x) \quad , \quad r'_1(u, x) = -\frac{j(u, x)}{\xi x} \quad , \quad r'_{2|3}(u, x) = \mp \frac{e^{-x\lambda_\mp} j(u, x)}{\lambda_\mp^2 (\lambda_+ - \lambda_-) x} , \quad (5.30)$$

whereas for $\xi = -4\mu^2$

$$f_0(u, x) = x (q_0(u) + r_0(u, x) + (q_1(u) + r_1(u, x))x + \{q_2(u) + r_2(u, x) + (q_3(u) + r_3(u, x))x\} e^{-2\mu x}) ,$$

$$r'_0(u, x) = -\frac{1 + \mu x}{4\mu^3 x} j(u, x) , \quad r'_1(u, x) = \frac{j(u, x)}{4\mu^2 x} , \quad r'_2(u, x) = \frac{(1 - \mu x) e^{2\mu x} j(u, x)}{4\mu^3 x} ,$$

$$r'_3(u, x) = \frac{e^{2\mu x} j(u, x)}{4\mu^2 x} , \quad (5.31)$$

where

$$j(u, x) := \left((\partial_x f_1(u, x)) \partial_x + f_1(u, x) \left(\partial_x^2 - \frac{\xi}{2\mu} \partial_x \right) - \frac{\xi}{\mu} \partial_x \partial_u + 2\partial_u \partial_x^2 \right) f_1(u, x) \quad . \quad (5.32)$$

6 Conclusion

To summarise, we have obtain all homogeneous spaces, solutions of the TMG, NMG and GMG field equations (with cosmological constant), at least formally. We classified them according to canonical representations of the structure constants obtained by Lorentz transformations. To obtain the explicit expression of the metric, we still have to solve an elementary algebraic problem, as this was explicitly demonstrated in subsection 4.2 for the type SII, which consists to find the linear transformation that maps the expressions of the structure tensor density and the structure vector obtained on their usual canonical expression, i.e. express the frame in the coordinate basis $(\theta^a[x^\mu])$ as linear combination of the canonical one or to directly integrate the expressions of the right invariant forms.

We also have determined the Petrov types of the traceless Ricci tensor, the Cotton–York tensor and the traceless $K_{\mu\nu}$ tensor of all Lorentzian 3-dimensional homogeneous geometries, which proves to be a useful tool to recognize equivalent solutions: for instance solutions of TMG, which are of Petrov type D, are biaxially squashed AdS_3 geometries [9].

In brief, we found that those of type *I* (which corresponds to all unimodular Bianchi types) allow all values of the cosmological constant; for types *II* (which could lead to the Bianchi types *II*, *VI*₀, *VII*₀ and *VIII*; for types *III* and *IV* (which correspond to Bianchi type *VI*₀ and *VIII*), in type *III* only the GMG has a solution of Petrov type *III* with negative cosmological constant. In case of homogeneous spaces corresponding to non-unimodular algebras, the Lorentzian classification of the algebra leads to solutions of types *T* and *SI* (which correspond to all non-unimodular types Bianchi types *III*, *IV*, *VI*_h and *VII*_h) which allow both signs of the cosmological constant; for type *SII* (which correspond

to Bianchi types IV , III and VI_h) with only negative values of the cosmological constant; for type $SIII$ (which correspond to Bianchi types III and VI_h) which allow both signs of the cosmological constant; for type L (which correspond to all non-unimodular Bianchi types, depending on the values of a and c) with positive cosmological constant for TMG & NMG and negative for the GMG; for type L' (which correspond to Bianchi type V for $b = 0$ and to III , VI_h for $b \neq 0$.)

In addition, we have also obtained the solutions of TMG, NMG and GMG field equation for VSI geometries (which imply a vanishing cosmological constant). The results are found to be of Petrov type III . They all contain, after having fixed the coordinate system, several arbitrary functions of a lightlike coordinate, reflecting the physical degrees of freedom of these solutions.

The above results provide a classification of all the solutions of massive gravity theories on CSI geometries. Of course not all of them can be considered as classical background geometries. For instance, it is well known that on Bianchi IX solutions, due to the compactness of the space, no global causal structure could be defined. It would be very interesting and instructive to study the perturbative stability of these backgrounds, and identify all the physically relevant configurations. It would be newsworthy to check if the Petrov type D solutions of NMG and GMG are biaxially squashed AdS_3 , like as in the TMG [9].

7 Acknowledgments

We would like to thank E. Bergshoeff, T. Damour, S. Detournay and U. Moschella for interesting and useful comments and discussions on various aspects of this work. This work has been supported by “Actions de recherche concertées (ARC)” de la Direction générale de l’Enseignement non obligatoire et de la Recherche scientifique Direction de la Recherche scientifique Communauté française de Belgique, and by IISN-Belgium (convention 4.4511.06). Ph.S. and K.S. reiterate their thanks to IHÉS and the University of Patras respectively, for hospitality where part of this work was developed.

A Petrov classification of homogeneous spaces

In this appendix we provide the Petrov classification of $S_{\alpha\beta}$, $C_{\alpha\beta}$, $\hat{K}_{\alpha\beta}$ tensors, for homogeneous spaces of types **A** and **B**, corresponding to unimodular and non-unimodular Lie algebras.

- Tables 1 and 2 correspond to traceless Ricci tensor for types **A** and **B** respectively.
- Tables 3 and 4 correspond to Cotton–York tensor for types **A** and **B** respectively.
- Tables 5 and 6 correspond to $\hat{K}_{\alpha\beta}$ tensor for types **A** and **B** respectively.

| Bianchi type A | Discriminant of the traceless Ricci tensor | Special values | Petrov type | Remarks |
|-----------------------|--|-----------------------------------|-------------|---|
| <i>I</i> | $\Delta_S = -\frac{1}{27}(a+2b)^2 c^2 ((a+b)^2 - c^2)^2 (a^2 - 4c^2)^2 \leq 0$ | | I_R | Generic |
| | | $c = 0, a \neq -b$ | D_t | |
| | | $c = 0, a = -b$ | O | $R^\alpha_\beta = -a^2/2 \delta^\alpha_\beta, AdS_3$ or flat spaces |
| | | $c = \pm(a+b)$ | D_s | |
| | | $a = -2b, b \neq c$ | D_t | If $b = c$, flat space |
| <i>II</i> | $\Delta_S = 0$ | $a = \pm 2c, b \neq \mp c \neq 0$ | D_s | It $b = \mp c$ or $c = 0$, flat space |
| | | | II | Generic |
| | | $b(a+b) = 0, a$ or $b \neq 0$ | N | If $a = b = 0$, flat space |
| <i>III</i> | $\Delta_S = 0$ | | III | Generic |
| | | $a = 0$ | N | |
| <i>IV</i> | $\Delta_S = \frac{1}{108}(a-b)^2(4\nu^2 - a^2)(-ab + b^2 + \nu^2)^2(-ab + b^2 + 4\nu^2)^2 \geq 0,$ | | I_C | Generic |
| | | $a = b$ | D_s | |

Table 1: Generic Petrov type of the traceless Ricci tensor on homogeneous space of class A and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type B | Discriminant of the traceless Ricci tensor | Special values | Petrov type | Remarks |
|-----------------------|--|-----------------------|------------------|--|
| <i>T</i> | $\Delta_S = -\frac{64}{27}b^6 (a^2 + k^2) (a^2 + k^2 - b^2)^2 \leq 0$ | | $I_{\mathbb{R}}$ | Generic |
| | | $k^2 = b^2 - a^2 > 0$ | D_s | |
| | | $b = 0$ | O | |
| <i>SI</i> | $\Delta_S = \frac{1}{108}(a+b)^6 (4k^2 - (a-b)^2) (k^2 + ab)^2$ | | I | $I_{\mathbb{C}}$ if $4k^2 > (a-b)^2$, $I_{\mathbb{R}}$ if $4k^2 < (a-b)^2$ |
| | | $k = \pm(a-b)/2$ | II | $a \neq -b$ |
| | | $a = -b$ | O | $R_{\beta}^{\alpha} = -2k^2\delta_{\beta}^{\alpha}$ (locally) AdS_3 |
| | | $k^2 = -ab$ | D | $ab < 0$; if $ b > a $, type D_s , if $ a > b $, type D_t |
| <i>SII</i> | $\Delta_S = 0$ | | N | Generic |
| | | $k = -a$ | O | |
| <i>SIII</i> | $\Delta_S = -\frac{1}{108}(4k^2 - a^2)(4\nu^2 - a^2)^3(k^2 - \nu^2)^2$, | | I | $I_{\mathbb{R}}$ if $k^2 > a^2/4$, $k^2 \neq \nu^2$, $I_{\mathbb{C}}$ if $k^2 < a^2/4$ |
| | | $k^2 = \nu^2$ | D | D_s if $k = \nu$, D_t if $k = -\nu$ |
| | | $k = \pm a/2$ | II | D_s if $a = 0$ |
| <i>L</i> | $\Delta_S = 0$ | | II | Generic |
| | | $c = 0$ | N | |
| | | | | If $b = 0$ or $b = -1$, flat space. |

Table 2: Generic Petrov type of the traceless Ricci tensor on homogeneous space of class B and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type A | Discriminant of the Cotton–York tensor | Special values | Petrov type | Remarks |
|-----------------------|--|---|------------------|--|
| <i>I</i> | $\Delta_C = -\frac{1}{1728}c^2((a+b)^2 - c^2)^2((3a^2 + 4c^2)^2 - (4ac - 8bc)^2)^2 \times$ $\times (a^2 - 4ab - 4(2b^2 + c^2))^2 \leq 0$ | | $I_{\mathbb{R}}$ | Generic |
| | | $c = 0, a \neq -b$ | D_t | |
| | | $c = 0, a = -b$ | O | AdS_3 or flat spaces |
| | | $c = \pm(a+b)$ | D_s | |
| | | $c = \pm \frac{1}{2}\sqrt{(a-2b)^2 - 12b^2}$ | D_t | $a \notin [-2(\sqrt{3}-1)b, 2(\sqrt{3}+1)b]$ |
| <i>II</i> | $\Delta_C = 0$ | $c = \pm \left(\frac{a}{2} - b \pm \sqrt{\frac{1}{2}(3b^2 - (a+b)^2)} \right)$ | D_s | $a \in [(\sqrt{3}-1)b, -(\sqrt{3}+1)b]$ |
| | | | <i>II</i> | Generic |
| | | $b(a+b) = 0, a \text{ or } b \neq 0$ | N | If $a = b = 0$, flat space |
| <i>III</i> | $\Delta_C = 0$ | | <i>III</i> | Generic |
| | | $a = 0$ | O | conformally flat, Ricci of type N |
| | | | | |
| <i>IV</i> | $\Delta_C = \frac{1}{6912} (4\nu^2 - a^2) ((a-b)(3a+b) - 4\nu^2)^2 (-ab + b^2 + \nu^2)^2 \times$ $\times (-3a^4 - 8a^3b + 2a^2b^2 + 9b^4 + 8(a^2 + 4ab - b^2)\nu^2 + 16b\nu^4)^2 \geq 0$ | | I_c | Generic |
| | | $4\nu^2 = (a-b)(3a+b)$ | D_s | |
| | | $a = -b = \nu$ | O | conformally flat, Ricci of type I_c |

Table 3: Generic Petrov type of the Cotton–York tensor on homogeneous space of class A and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type B | Discriminant of the Cotton–York tensor | Special values | Petrov type | Remarks |
|-----------------------|---|---|------------------|---|
| <i>T</i> | $\Delta_C = -\frac{64}{27}b^6(a^2 - b^2 + k^2)^2(4a^2 - b^2 + k^2)^2 \times$ $\times \left((2a^2 + b^2)^2 + k^2(5a^2 - 2b^2) + k^4 \right) \leq 0$ | $b = 0$ | $I_{\mathbb{R}}$ | Generic |
| | | $k^2 = b^2 - a^2$ | O | dS_3 |
| | | $k^2 = b^2 - 4a^2$ | D_s | |
| | | $k^2 = b^2, a = 0$ | D_s | |
| | | | O | Conformally flat, Ricci type D_s |
| <i>SI</i> | $\Delta_C = -\frac{1}{6912}(a+b)^6(ab+k^2)^2((a-3b)(3a-b)-4k^2)^2 \times$ $\left((3(a-b)^2 + 4ab)^2 - 4k^2(3(a-b)^2 - 8ab) + 16k^4 \right)$ | $a = -b$ | I | If $(...) > 0$ $I_{\mathbb{R}}$ and if $(...) < 0$ $I_{\mathbb{C}}$ |
| | | $k^2 = -ab$ | O | AdS_3 |
| | | $4k^2 = (a-3b)(3a-b)$ | D | $ab < 0, D_s$ if $ b > a , D_t$ if $ a > b $ |
| | | $16k^2 = 3(a-b) \pm \sqrt{-(3a+b)(a+3b)}$ | D | $a \notin [b/3, 3b], D_s$ if $a \in [-b, 3b], D_t$ if $a \notin [-b, 3b]$ |
| | | | II | O if $b = -a$ and $k = a^2$ or $4a^2$ |
| <i>SII</i> | $\Delta_C = 0$ | | N | Generic |
| | | $a = -k/2$ | O | Ricci type N |
| | | $a = -k$ | O | AdS_3 |
| <i>SIII</i> | $\Delta_C = \frac{1}{6912}(4\nu^2 - a^2)^3(k^2 - \nu^2)^2(4\nu^2 - 4k^2 + 3a^2)^2 \times$ $\times (9a^4 + 16(k^2 - \nu^2)^2 - 12a^2(k^2 + 2\nu^2))$ | $k^2 = \nu^2$ | I | If $(...) < 0$ $I_{\mathbb{R}}$ and if $(...) > 0$ $I_{\mathbb{C}}$ |
| | | $4k^2 = 4\nu^2 + 3a^2$ | D | D_s if $k = \nu, D_t$ if $k = -\nu$ |
| | | | D | D_s if $\nu k > 0, D_t$ if $\nu k < 0$ |
| | | $16k^2 = 3a \pm \sqrt{16\nu^2 - 3a^2}$ | II | O if $a = 0$ and $k^2 = \nu^2$ |
| <i>L</i> | $\Delta_C = 0$ | | II | Generic |
| | | $c = 0$ | O | Conformally flat, Ricci type N |

Table 4: Generic Petrov type of the Cotton–York tensor on homogeneous space of class B and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type A | Discriminant of the $\hat{K}_{\alpha\beta}$ tensor | Special values | Petrov type | Remarks |
|-----------------------|--|--|------------------|---|
| <i>I</i> | $\Delta_{\hat{K}} = \dots \leq 0$, given in (A.1) | | $I_{\mathbb{R}}$ | Generic |
| | | $c = 0, (4b + 21a)(b + a) \neq 0$ | D_t | |
| | | $c = 0, (4b + 21a)(b + a) = 0$ | O | for $b = -a$ AdS_3 or flat spaces |
| | | $c = \pm(a + b)$ | D_s | |
| | | $c = \pm \frac{\sqrt{-5a^2+2a^2b-40ab^2-64b^3}}{2\sqrt{5a+26b}}$ | D_t | $O : (a + 2b)(25a + 42b)(5a^3 + 36a^2b - 44ab^2 + 64b^3) = 0$ |
| <i>II</i> | $\Delta_{\hat{K}} = 0$ | $21a^3 + 4ac(\pm 6b + 5c) + a^2(4b \pm 26c) \pm 8c(8b^2 \pm 8bc + 5c^2) = 0$ | D_s | |
| | | | II | Generic |
| | | $(4b + 21a)(b + a) = 0, a$ or $b \neq 0$ | N | If $a = b = 0$, flat space |
| <i>III</i> | $\Delta_{\hat{K}} = 0$ | | III | Generic |
| | | $a = 0$ | O | conformally flat, Ricci of type N |
| | | | | |
| <i>IV</i> | $\Delta_{\hat{K}} = \dots \geq 0$, given in (A.4) | | I_c | Generic |
| | | $-21a^3 + 15a^2b + ab^2 + 5b^3 + 4(13a - 5b)\nu^2 = 0$ | D_s | |

Table 5: Generic Petrov type of the $\hat{K}_{\alpha\beta}$ tensor on homogeneous space of class A and particular values of the structure tensor density, leading to special Petrov types.

| Bianchi type B | Discriminant of the $\hat{K}_{\alpha\beta}$ tensor | Special values | Petrov type | Remarks |
|-----------------------|---|--|--------------------|---|
| <i>T</i> | $\Delta_{\hat{K}} = \dots \leq 0$, given in (A.5) | $b = 0$ | $I_{\mathbb{R}}$ | Generic |
| | | $k^2 = b^2 - a^2$ | O | dS_3 |
| | | $k^2 = -24a^2 + 5b^2 \pm 16\sqrt{2} a \sqrt{a^2 - b^2}$ | D_s | |
| | | | D_s | |
| <i>SI</i> | $\Delta_{\hat{K}} = -\#((a-b)^2(21a^2 + 10ab + 21b^2)^2 - 4(63a^4 - 348a^3b + 970a^2b^2 - 348ab^3 + 63b^4)k^2 + 16(39a^2 - 118ab + 38b^2)k^4 - 64k^6)$, $\# \geq 0$, given in (A.6) | | I | If (...) > 0 $I_{\mathbb{R}}$ and if (...) < 0 $I_{\mathbb{C}}$ |
| | | $a = -b$ | O | AdS_3 |
| | | $k^2 = -ab$ | D | $ab < 0$, D_s if $ b > a $ and D_t if $ a > b $ |
| | | $4k^2 = 19(a-b)^2 - 20ab \pm 16 a-b \sqrt{a^2 - 6ab + b^2}$ (...) $= 0$ | $D_{t,s}$ D_t | $(a+b)(5a-3b \pm s[a-b]\sqrt{a^2 + b^2 - 6ab})$ |
| <i>SII</i> | $\Delta_{\hat{K}} = 0$ | | N | Generic |
| | | $4a = k(-2 \pm \sqrt{2})$ | O | Ricci type N |
| | | $a = -k$ | O | AdS_3 |
| <i>SIII</i> | $\Delta_{\hat{K}} = \#(441a^6 - 84a^4(3k^2 + 26\nu^2) - 64(k^3 - 5k\nu^2)^2 + 16a^2(39k^4 - 24k^2\nu^2 + 169\nu^4))$, $\# \geq 0$, given in (A.8) | | I | If (...) < 0 $I_{\mathbb{R}}$ and if (...) > 0 $I_{\mathbb{C}}$ |
| | | $k^2 = \nu^2$ | D | D_s if $k = \nu$, D_t if $k = -\nu$ |
| | | $4k^2 = 19a^2 + 20\nu^2 \pm 16 a \sqrt{a^2 + 4\nu^2}$ | D | D_s if $\nu k > 0$, D_t if $\nu k < 0$ |
| | | (...) $= 0$ | II | O if $a = 0$ and $k^2 = 5\nu^2$ |
| <i>L</i> | $\Delta_{\hat{K}} = 0$ | | II | Generic |
| | | $c = 0$ | O | Conformally flat, Ricci type N |

Table 6: Generic Petrov type of the $\hat{K}_{\alpha\beta}$ tensor on homogeneous space of class B and particular values of the structure tensor density, leading to special Petrov types.

Finally, we give the expressions of the discriminants of $\widehat{K}_{\alpha\beta}$ tensor for all the cases

$$\begin{aligned}\Delta_{\widehat{K}}^I &= -\frac{1}{110592}((a+b)^2 - c^2)^2 c^2 (5a^3 - 2a^2b + 40ab^2 + 64b^3 + 4(5a + 26b)c^2)^2 \times \\ &\times (21a^3 + 4ac(-6b + 5c) + a^2(4b + 26c) + 8c(8b^2 - 8bc + 5c^2))^2 \times \\ &\times (21a^3 + a^2(4b - 26c) + 4ac(6b + 5c) - 8c(8b^2 + 8bc + 5c^2))^2 \leq 0 \quad ,\end{aligned}\tag{A.1}$$

$$\Delta_{\widehat{K}}^{II} = 0 \quad ,\tag{A.2}$$

$$\Delta_{\widehat{K}}^{III} = 0 \quad ,\tag{A.3}$$

$$\begin{aligned}\Delta_{\widehat{K}}^{IV} &= \frac{1}{442368} (4\nu^2 - a^2) (-ab + b^2 + \nu^2)^2 (-21a^3 + 15a^2b + ab^2 + 5b^3 + 4(13a - 5b)\nu^2)^2 \times \\ &(105a^6 + 76a^5b + 381a^4b^2 - 160a^3b^3 - 45a^2b^4 + 84ab^5 - 441b^6 - 4(171a^4 + 56a^3b + 486a^2b^2 - 160ab^3 - \\ &- 41b^4)\nu^2 + 16(91a^2 - 20ab + 105b^2)\nu^4 - 1600\nu^6)^2 \geq 0 \quad ,\end{aligned}\tag{A.4}$$

$$\begin{aligned}\Delta_{\widehat{K}}^T &= -\frac{64}{27}b^6(a^2 - b^2 + k^2)^2 \left((8a^3 + 13ab^2)^2 + (112a^4 - 74a^2b^2 + 25b^4)k^2 + (49a^2 - 10b^2)k^4 + k^6 \right) \times \\ &\times \left(64a^4 + (-5b^2 + k^2)^2 + 16a^2(b^2 + 3k^2) \right)^2 \leq 0 \quad ,\end{aligned}\tag{A.5}$$

$$\begin{aligned}\Delta_{\widehat{K}}^{SI} &= -\frac{1}{442368}(a+b)^6(ab+k^2)^2(105a^4 - 156a^3b + 502a^2b^2 - 156ab^3 + 105b^4 - 8(19a^2 - 58ab + 19b^2)k^2 + \\ &+ 16k^4)^2 \left((a-b)^2(21a^2 + 10ab + 21b^2)^2 - 4(63a^4 - 348a^3b + 970a^2b^2 - 348ab^3 + 63b^4)k^2 + \right. \\ &\left. 16(39a^2 - 118ab + 39b^2)k^4 - 64k^6 \right) \quad ,\end{aligned}\tag{A.6}$$

$$\Delta_{\widehat{K}}^{SII} = 0 \quad ,\tag{A.7}$$

$$\begin{aligned}\Delta_{\widehat{K}}^{SIII} &= \frac{1}{442368} (4\nu^2 - a^2)^3 (k^2 - \nu^2)^2 \left(105a^4 + 16(k^2 - 5\nu^2)^2 - 8a^2(19k^2 + 33\nu^2) \right)^2 \times \\ &\times \left(441a^6 - 84a^4(3k^2 + 26\nu^2) - 64(k^3 - 5k\nu^2)^2 + 16a^2(39k^4 - 24k^2\nu^2 + 169\nu^4) \right) \quad ,\end{aligned}\tag{A.8}$$

$$\Delta_{\widehat{K}}^L = 0 \quad .\tag{A.9}$$

References

- [1] S. Deser, R. Jackiw and G. t Hooft, *Annals Phys.* **152** (1984) 220.
- [2] S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975.
- [3] S. Deser, R. Jackiw and S. Templeton, *Annals Phys.* **140** (1982) 372 [Erratum-ibid. **185** (1988) 406]
[*Annals Phys.* **185** (1988) 406] [*Annals Phys.* **281** (2000) 409].
- [4] M. Nakasone and I. Oda, *Prog. Theor. Phys.* **121** (2009) 1389 [arXiv:0902.3531 [hep-th]].
- [5] S. Deser, *Phys. Rev. Lett.* **103** (2009) 101302 [arXiv:0904.4473 [hep-th]].
- [6] I. Oda, *JHEP* **0905** (2009) 064 [arXiv:0904.2833 [hep-th]].
- [7] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Phys. Rev. Lett.* **102** (2009) 201301
[arXiv:0901.1766 [hep-th]].
- [8] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Phys. Rev. D* **79** (2009) 124042 [arXiv:0905.1259
[hep-th]].
- [9] D. D. K. Chow, C. N. Pope and E. Sezgin, *Class. Quant. Grav.* **27** (2010) 105001 [arXiv:0906.3559
[hep-th]].
- [10] D. D. K. Chow, C. N. Pope and E. Sezgin, *Class. Quant. Grav.* **27** (2010) 105002 [arXiv:0912.3438
[hep-th]].
- [11] W. Kundt, *Z. Phys.* **163**, 77 (1961).
- [12] W. Kundt, M. Trümper, *Akad. Wiss. Lit. Mainz, Abhandl. Math.-Nat. Kl.* **12** (1962).
- [13] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, *Exact Solutions of Einstein's
Field Equations Second Edition*, Cambridge University Press, Cambridge (2003).
- [14] M. E. Ortiz, *Annals Phys.* **200** (1990) 345.
- [15] G. Moutsopoulos, [arXiv:1211.2581 [gr-qc]].
- [16] A. Coley, S. Hervik and N. Pelavas, *Class. Quant. Grav.* **25** (2008) 025008 [arXiv:0710.3903 [gr-qc]].
- [17] Y. Liu and Y. W. Sun, *Phys. Rev. D* **79** (2009) 126001 [arXiv:0904.0403 [hep-th]].
- [18] I. Ozsvath, *J. Math. Phys.* **6**, 4 (1964) 590-609.
- [19] J. Plebanski, *Acta Phys. polon.* **26** (1964) 963.

- [20] R. E. Hiromoto and I. Ozsvath, *Gen. Rel. Grav.* **9** (1978) 299.
- [21] S. Hawking, G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge (1973).
- [22] G. Gibbons, C. Pope, and E. Sezgin, *Class. Quant. Grav.* **25** (2008) 205005, [arXiv:0807.2613 [hep-th]].
- [23] Ph. Spindel, “Gravity before Supergravity”, in *Supersymmetry*, ed. by K. Dietz, R. Flume, G. v. Gehlen and V. Rittenberg, Proceedings of NATO Advanced Study Institute , **125 B** Bonn 1984 (Plenum, New York, 1984), pp 455-533.
- [24] A. Coley, S. Hervik and N. Pelavas, *Class. Quant. Grav.* **23** (2006) 3053 [gr-qc/0509113].